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The Abstract and the Concrete in the Development of School Geometry*

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I. THE MENTAL DEVELOPMENTS OF EUCLIDEAN GEOMETRY

A. *In Euclid's Elements*

ABOUT 300 B.C. the Greek mathematician and philosopher, Euclid, whose influence was felt in Athens and Alexandria, wrote his famous work in 13 volumes, known throughout the world as "The Elements." It is known that this work is considered as an example of logical power in mathematics, since it is developed (gradually) from a definite number of definition through provable statements, or postulates, then to fundamental truths or axioms, and thereby spread out finally into propositions. The synthetic proofs of these propositions follow strictly the scheme:

1. Proposition
2. Assumption
3. Generalization
4. Proof

Next to the Bible, "The Elements" is the most widely used book in the entire world. It is known that it ("The Elements") appeared in close to 1700 different editions in all cultured languages. It is also known that this Geometry represented the material for the education of the youth in geometrical thinking up to the latest times, until about 30 years ago, when the International Commission on Mathematical Education suggested the development of the geometry subject matter along psychological and practical lines.

For almost 2,000 years Euclidean geometry was valid and was considered as the only kind possible until the latest geometric methods of thinking were developed. Before we examine these movements which led to reforms in Euclidean geometry whether as a science or a school subject we shall consider some introductory statements from Euclid's System of Axioms, namely: definitions, postulates, and axioms. I quote from the English edition of Simson:

BOOK I

Definitions (S.1)

I. A Point is that which has no parts or which has no magnitude.
II. A line is length without breadth.
III. The extremities of a line are points.
IV. A straight line is that which lies evenly between its extreme points.
V. A superficies (area) is that which has only length and breadth.
VI. The extremities of a superficies are lines.
VII. A plane superficies is that, in which any two points being taken, the straight line between them lies wholly in that superficies.

. . . .

* Read before the third Mathematics Conference at Teachers College, Columbia University, July 23, 1936.

XXXV. Parallel straight lines are such as are in the same plane and which, being produced ever so far both ways, do not meet.

Postulates

I. Let it be granted that a straight line may be drawn from any one point to any other point.

IV. All right angles are equal to one another.

V. If a straight line meets two straight lines, so as to make the two interior angles on the same side of it, taken together less than two right angles, these straight lines being continually produced, shall at length meet upon that side, on which are the angles, which are less than two right angles.

Axioms

II. If equals be added to equals, the wholes are equal.

Proposition 27

If a straight line falling upon two other straight lines makes the alternate angles equal to one another, these 2 straight lines shall be parallel.

Proposition 32

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles . . . and the three interior angles of every triangle are equal to two right angles.

Book III
Definitions

II. A straight line is said to touch a circle, when it meets the circle, and being produced does not cut it. (A straight line touches a circle in *one* point.)

III. Circles are said to touch one another, which meet, but do not cut one another. (Circles touch in *one point.*)

B. *Non-Euclidean Geometry*

Euclid's Elements was a classical work, but it was not easily understood by his contemporaries. Consequently there was no lack of commentaries, among whom *Proclus* (410–485 A.D.) occupied a significant place. This Proclus already pointed out that the so-called Fifth Postulate of Book I, known as the Axiom of Parallels, cannot be thoroughly understood, and he suggested that this Postulate must be proved. He finally indicated that in this Geometry it is possible to draw through a (given) point only one parallel to a (given) straight line.

For about 1000 years this problem lay dormant, until the Catholic priest, *Gerolamo Saccheri* (1667–1733) and the great mathematician and architect *Johann Heinrich Lambert* (1728–1777), a protegé of Frederick the Great, began to improve the logical system of Euclid.

They attempted to give a proof of the Fifth Postulate, but they discovered that in order to prove this proposition it was necessary to prove another proposition which in turn made use of the Fifth Postulate in the process of its proof.

The solution of this problem was offered in 1832 by two Hungarian mathematicians:

1. Wolfgang Bolyai (1775–1856) and his son
2. Johann Bolyai (1802–1860)

They employed the method of assuming the unconventional which is often commonly employed in philosophy. While in Euclidean geometry the sum of the angles of a triangle was 180° and to a straight line only one parallel line could be drawn, the two Bolyai set out to develop a geometry in which two parallel lines could be drawn and the sum of the angles of a triangle was less than 180°. They succeeded in formulating such a geometry.

How well could the Russian mathematician, *Nicolaus Ivanovitch Lobatschewsky* (1793–1856) assist them productively in this problem is immaterial for us, except that the first arrived at the same idea either in 1829 or in 1840. We must note the fact, however, that he was working on this problem also.

Another distinct step forward was made by the researches of *Bernhard Riemann* (1826–1866) who proposed a new non-Euclidean geometry in his dissertation: "Concerning the hypotheses that are fundamental to geometry."[1] His geometry has the following properties:

[1] D. E. Smith, "A Source Book in Mathematics." McGraw-Hill Company, New York, 1929, p. 411 f.

1. All lines are finite.

2. No parallel can be drawn through a point to a given line.

3. The sum of the angles of a triangle is greater than 180°.

In this case it concerns with a geometry closely related to spherical trigonometry. If we consider that two parallel straight lines have a point of intersection at infinity, we can combine the 3 geometries as follows:

1. The Bolyai-Lobatschewsky: the hyperbolic geometry with two points at infinity and with the sum of the angles of a triangle less than 180°. Through a point two parallels can be drawn to a straight line.

2. The Euclidean or parabolic geometry: one point at infinity; the sum of the angles of a triangle is equal to 180°; through a point one parallel can be drawn to a straight line.

3. The Riemannian or elliptic geometry: no points at infinity; the sum of the angles of a triangle is greater than 180°; no parallel can be drawn to a straight line.

The three dimensional Euclidean geometry is thus a special case of the general geometries. In this the fifth postulate is valid.

But the greatest influence on the future was produced by Riemann's dissertation; in it he pointed out that we do not need any definition of a point and thereby any other fundamentals in mathematics except that it suffices that we know the following: a point is determined in the Euclidean space by three coordinates, x y and z. In the succeeding years it became common to say: *x, y and z is a point.*

C. *Hilbert's Foundations*

While up to this time considerations were centered on the fifth postulate and thus one only fundamental proposition of the Euclidean System, in the succeeding years attention was given to the entire system. It was found that it cannot be logically vigorous if it does not fulfill the following conditions:

1. The system of axioms is not complete.

2. It contains no contradictions.

3. The fundamental assumptions are not independent of one another.

On this basis, after considering Riemann's investigations, as well as those of his successors, the Göttingen scientist, *David Hilbert*, formulated in 1899 a new system of geometry, known now as: Hilbert, *Grundlagen der Geometrie*, 7th edition, Teubner, 1930.[2] In this system, the point, straight line, and plane play no part, except through the meaning the relations of these objects are developed by means of words: lies, between, parallel, congruent, valid, etc. And on these considerations he bases his new system which through the aid of the further investigations of the Americans an independent structure of geometry was developed.

D. *Projective Geometry*

While by means of these highly spiritual works the Greek geometry was fundamentally preserved, in the 15th century a new geometry was developed. At the end of the 14th century, in Italy, arose the problem: how could the painter represent spatial objects just as they are seen. This problem was solved by a circle of artists in Florence at center of which was the famous architect and painter, *Philippo Brunelleschi* (1377–1446). To this circle belonged the profound thinker and humanist and versatile artist *Leon Battista Alberti* (1404–1472). They discovered the fundamental laws of perspective as well as the projection method, that of descriptive geometry. It is of great interest that the first printed work of perspective by Alberti was published in 1435–1436, and last week we celebrated on the 12 of July the 500th jubilee of the publication of the Italian edition of this work.[3] We

[2] Hilbert, "Foundations of Geometry," Open Court Publishing Company, Chicago, 1902.

[3] In Latin on August 26, 1435, in Italian on July 17, 1436. *Hubert Janitschek:* Leon Battista Alberti, Kleinere kunsttheoretische Schriften, W. Braumüller, Wien, 1877; G. Wolff: "500 Jahre Perspektive." Unterrichtsblätter f. Mathematik und Naturwissenschafter, 1936, p. 235f; Hieronijmo Mancini; Leonis Baptista Alberti: Opera Inedita, Florentia, 1890.

shall content ourselves with mentioning the names of the principal investigators who contributed to this science:

Girard Desargues (1590–1662)
Gaspard Monge (1746–1818)
Jean Victor Poncelet (1789–1867)
Jacob Steiner (1796–1863)
Christian von Staudt (1798–1868)[4]

It will be sufficient to present briefly the main characteristics of this new geometry.

1. It fixes the first line with the three dimensional space, and the second line with the plane; in this space, there is to be pictured or in which it is projected.

2. For picturing we employ the practical method of drawing.

3. The geometrical investigations are conducted by means of motions, the study of functional independence as well as by means of Dynamics.

4. The parallel axiom is absent in projective geometry, although from it directly comes the infinitely distant element (Desargues).

5. Properties of figures may be obtained by means of transformations: for example, from equilateral triangle to tetrahedrons; from square to cube; from the range to the pencil; from circle to ellipse; pole-polar; quadrilateral-complete quadrilateral; Pascal-Brianchon (Principle of Duality, Reciprocity).

6. In contrast to Euclidean Geometry we do not go from special cases to general ones—but from general to special ones. Hyperbola, parabola, ellipse, circle, straight line, point are not special figures; they are considered generally as conic sections or central projections.

But other methods of representation must be employed in the process of investigation: displacement, translation, rotation and cuts serve the purpose of proofs; the transition from one figure to another must be gradual.

We thus see that this is a new method of thinking, quite foreign to the Greek geometry. *Leibniz* (1646–1716) and especially *Poncelet* spoke of a new principle of development, the principle of continuity in geometry parallel to the principle of continuity in Function Theory.

7. *Felix Klein* (1849–1925) went one step farther and examined the common properties of the methods of representation. In the famous Erlanger Program (1871) he connected these geometries according their properties of invariance and thereby according the point of view of group theory. The school geometry falls under the *projective group*. It embraces:

1. The perspective or central collineation
2. Parallel projection or affinity
3. Similarity
4. Symmetry
5. Congruence

8. *Cayley* (1821–1895) rightly recognized the methods of projective geometry when he said: "Projective geometry is all geometry"—

II. The Influence of Pedagogy on School Geometry

A. *Pedagogical Movements in Germany*

In contrast to the methods of projective geometry which is distinguished by: observation, practical drawings, motion and functional independence through the principle of motion, Euclidean geometry was based on pure thinking which quite often needed figures for proofs. The representation was for the young minds of pupils very difficult; it never called forth his interests; it limited itself to the plane, or to two dimensions, and it required little space perception. Practical applications were lacking, and the cultural value of geometry was never called to the pupils' attention. That the pupils were not enthusiastic with the Elements is testified by diaries and memoirs of great men. What they really thought about geometry can be seen from verses taken from a Euclid owned by a pupil:

[4] G. Wolff: "Die Entwicklung der Abbildungsidee in Wissenschaft und Schule." Unterrichtsblätter für Mathematik und Naturwissenschaften, 1933, p. 296 ff. and p. 332 ff.

If there should be another flood
Hither for refuge fly,
Were the whole world submerged,
This book would still be dry.

Aroused by projective geometry and excited by pure pedagogical and psychological considerations, *Pestalozzi* (1746–1827), *Herbart* (1776–1841), and *Fröbel* (1782–1852) attempted to change the teaching of geometry. It was directed toward the play-activity of the pupil, so that by self-aroused interest the child might be excited. Besides they considered the so-called biogenetic laws according to which the development of every man (Antroprogeny) follows the ladder of evolution (Phylogeny). Thus the teaching of geometry had to be completely changed. Instead of rigorous logic empirical observation was intended to take its place. Instead of pure mathematics applied mathematics was to replace it, as for example, it was used in a workshop.

Through the help of the school mathematicians these ideas were made practical; in Germany this took place at first in about 1870. At that time a school mathematician *Hubert Müller* wrote a geometry textbook *along the ideas of projective geometry*. But it took a long time until these ideas were carried through.[5]

B. *Perry in England*[6]

In other countries similar movements took place; of all, in England about 1900. There they desired to improve the preparation of engineers. And thus, at the meeting of the English Association for the Advancement of Science in Glasgow in 1900, Professor John Perry proposed that they break with the non-modern geometrical training in the Euclidean sense—and to search for a subject matter content that would arouse the children spiritually and give play to their individual activities. His ideas were reduced to the statement he made:

[5] Hubert Müller: "Leitfaden der ebenen Geometrie." 1874 ff. Teubner, Leipzig.

[6] G. Wolff: "Der Mathematische Unterricht der höh. Knabenschulen in England," Teubner, 1915, pp. 67 ff.

"I am a practical man, gentlemen." These ideas were made still more fundamental although since then the geometry teaching in the English schools was still done from English translations of old editions of Euclid, and even 12 years later one could find in a great number of English schools the traditional copies of "Euclid"!!

C. *The International Commission on the Teaching of Mathematics*

The reorganization in England and in other countries was finally made real after the total realization of these ideas under the leadership of Felix Klein[7] in Germany and of David Eugene Smith in America.[8] This reform became quite apparent in the improvement of teaching of algebra, especially by the introduction of Differential and Integral Calculus in the secondary schools. But very little was required of geometry as far as a space perception is concerned but it was suggested that it follows the lines of descriptive geometry.

In Germany, this reform movement continued quietly since 1916 by means of which the teaching of geometry finally complied with the above requirements. It was carried through along the lines of the ideas of transformations of group theory and of descriptive geometry. As a consequence we arrange the subject matter in the sense of collineation groups.

D. *Mathematics as an Empirical Natural Science*

When nowadays one studies geometry in the sense of motion, in the sense of measurement, in the sense of drawing, this can be done only after the fundamentals of projective geometry are introduced in schools. But thereby two important requirements of education are fulfilled:

[7] Klein-Schimmack: "Der Mathematische Unterricht an den höheren Schulen," Teubner, Leipzig, 1907.

[8] David Eugene Smith, "Anschauung und Experiment im Mathematischen Unterricht der höheren Schulen." Berichte und Mitteilungen der IMUK, Heft VIII, B. G. Teubner, 1913, p. 164 f.

1. The individual activities of the pupils are placed in the foreground.

2. Geometry as a natural science such as in physics is studied through experimentation and through deductive considerations. We measure line-segments and angles and search for relations, that are to be established between them. This is the method employed by natural scientists.

We know, however, that natural laws cannot be expressed exactly but only by approximation. The famous formula of Galileo Galilei (1564–1642)

$$S = \tfrac{1}{2}gt^2$$

is mathematically rigourously exact. But every experimentator knows that he must definitly prepare his experiment when he wishes to obtain exact numerical values. And I will not say anything new to a physicist when I tell him the laws of volumes of gases or the laws of electricity must be very carefully studied experimentally in order to avoid the obtaining of rough values; these will seldom be exact.

Poincaré again pointed out this difficulty in his philosophical work, "Science and Hypotheses," and also in "The Value of Science." Pure mathematics in order to be applied to facts must be used in the sense of application of formulas of natural laws to experiments.

Let me give a few illustrations of these fundamental facts from pure and applied mathematics.

1. We know the approximate value

$$\sqrt{2} = 1.4142 \cdots$$

We know that we can write

$$\sqrt{2} \sim 1.4,$$

$$\sqrt{2} \sim 1.41,$$

$$\sqrt{2} \sim 1.414.$$

But these represent rough values of $\sqrt{2}$. Only the limiting value with an infinite number of places gives the true value.

2. When we calculate $\overline{4.02}^2$, we know that $\overline{4.01}^2 = 16.0801$. In practice we write it as follows: $(4+0.01)^2 = 16+8.0.01+0.01^2$.

But the last term on the right, 0.01^2 is very small. Every computer or worker is satisfied with the "*First approximation*": $4.01^2 \sim 16.08$.

3. $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad [x < 1].$$

a. The first approximation is

$$\frac{1}{1-x} \sim 1 + x$$

b. The second approximation is

$$\frac{1}{1-x} \sim 1 + x + x^2$$

c. The third approximation is

$$\frac{1}{1-x} \sim 1 + x + x^2 + x^3$$

If $x = \frac{1}{2}$, we have

a. $\frac{1}{1-x} \sim 1.5,$

b. $\frac{1}{1-x} \sim 1.75,$

c. $\frac{1}{1-x} \sim 1.875.$

What is so obvious in these numerical examples should only be further carried out in geometry. In many cases must the geometer work with approximations, when he wishes to apply geometry practically. When we mark the vertices of a hexagon with compasses on a circle, we always note that our compasses never give exactly a hexagon.

Thus far we have tirelessly tried to develop *a priori* the notions of points and straight lines when these could be obtained *a posteriori* and that should have been our problem. We do not need to per-

form mental acrobatics, we need to study only reality. We should be interested not in the mathematical point but in the empirical point, not in the ideal straight line but in the material straight line, not in the abstract plane but in the real plane.

The formulations of propositions by Euclid gives limiting values, in fact, however, we have approximations to reality.

III. Practical Achievements of Euclidean Geometry

A. *Ancient Times*

In the Metaphysics of *Aristotle* (384–322 B.C.) (Vol. 2, 997), we find the following statement, which we translate freely:

"It is not also true that geodesy deals with observable and passing magnitudes; because if these were passing when they have to do with "the passing." But truly, astronomy also has nothing to do with perceptible magnitudes, because it has to do with the sky. Neither the perceptible lines and curves are such in reality as the geometer defines them, nor the ruler touches the circle in one point but it is as *Protagoras* said in refuting the geometers, because the perceptible is neither so straight nor so curved as the mathematicians conceive them[9] and also *Sixtus Impericus* (200–250 A.D.) as a sceptic pointed out in his adversus mathematicos the difficulties between theories and applications of geometry.

B. *Middle Ages*

More definite is the distinction between theory and practice as pointed out by Leon Battista Alberti already mentioned. In his work of 1435–1436 written not for mathematicians but for painters he said, "*The point is a material magnitude.* A line should be thought of as some sort of strip (band). And a plane is visually represented as a piece of linen. The circle is represented by a ring." The mathematicians of that time were wholly captivated by the Euclidean representation, and they attacked Alberti. Alberti, in a special paper, "De punctis et lineis apud pictores"[10] defended himself against these attacks, and he pointed out that he conceived a point as a picturization of a body of some kind of an atom because a painter cannot deal with it in any other manner, moreover, he must be able to symbolize a line and a plane. Still more definitely he expressed himself in his work, "Elementa picturea"[11] where on page 51 he said:

1. A point is a trace akin to an atom; so small it is that it can only be made by hand.
2. Lines are traces drawn from point to point, as thin as a border of a plane drawn by a painter.
3. A plane is the boundary of a body.

C. *Modern Times*

Only in modern times was approach made to such ideas (as above). In reality, the fact that a point, a line, and a plane are material objects in teaching introduces certain difficulties in the application of the propositions discovered by pure logic.

1. We know that two straight lines intersect in one point; when the point of intersection subtends less than a right angle the angle is acute, the two straight lines may have some common space.
2. When we measure the angles of a triangle, we roughly find that the sum of the angles is not exactly 180°, but it deviates somewhat from this exact value.
3. The measure of length (space) with the precision as demanded by exact geometry is impossible by means of technical appliances.
4. We learn that tangents touch a circle in one point. We know, however, that these tangents have roughly common lengths with the circles.
5. When two circles are tangent to one another they should have a common tangent, and all these three should touch in

[9] Protagoras (480–410 B.C.). From him comes the statement: Man is the measure of all things.

[10] Opera Inedita, p. 66.

[11] Opera Inedita, p. 48 f.

the same point. This is seldom achieved in practice.

6. According to Euclid a straight line is determined by two points. In practice, however, a straight line is drawn after three points are obtained.

Such facts had *Helmholtz* (1821–1894) to examine empirical geometry closely, and when he categorically agreed with the last statement he certainly was compelled to agree with all other examples.[12]

We may add to the above examples another classical one given by the positivist, John Stuart Mill (1806–1873): "There is no circle whose radii are all equal."[13]

Departing from such considerations *Moritz Pasch* attempted to formulate a geometry that would have the character of a natural science. In the preface he said:[14]

> The multitude of applications to natural sciences which geometry is capable of in practical life, rests mainly on the fact that geometrical concepts closely represent empirical objects, and when they are gradually abridged by artful ideas (concepts) in order to demand theoretical development and thereby when we limit ourselves to empirical procedure it remains for geometry to attain the character of a natural science.

From *a priori* and *a posteriori* considerations, Felix Klein in his "Elementarmathematik vom höheren Standpunkte aus" Vol. III,[15] p. 5 ff. made the significant distinction between precision mathematics and mathematics of approximations, and this radically influenced mathematical thinking:

> Next we shall consider the character of practical geometry. It is that geometry with which we have to deal when we want to learn concrete spatial relationships, thus in drawing, modelling, as well as in measuring.—Here, all operations are only approximate to some degree. Here in this branch we must set forth definitions, and axioms in the following manner:
>
> A point in this branch is a body of such small dimensions that we disregard them.
>
> A curve, especially a straight line, is some kind of a trace whose breadth is insignificant in comparison with its length.
>
> Two points determine a straight line joining them more definitely the farther apart they are from one another. If they are close to one another then they are of less value in this determination.
>
> Two intersecting lines determine one point more definitely the closer the angle between these two straight lines is to a right angle. The smaller the angle the less proper is the determination of the point of intersection."

Also in America such an influential investigator as *George Birkhoff* of Cambridge, Mass. attempted to formulate a visual mathematics of the type useful for school children.[16]

In the 4th Yearbook of the National Council of Teachers of Mathematics for 1929 on page 118 there is reference to the *Geometry of Reality* by Professor J. Hjelmslev of Copenhagen:

> The first is a science of things-edges of rulers, table tops,—considered empirically and inductively, its theorems being only partly proved.
>
> Pedagogically the geometry of reality has the advantage of treating of real things but its theorems lack the simpleness of the abstract geometry. The modern teaching of geometry, with the emphasis on intuition as a basis, is, properly speaking, a union of the two. As with all textbooks of a revolutionary character, those of Hjelmslev have had difficulties in obtaining access to schools.

This Geometry is based on the above developed ideas, and Hjelmslev developed in one textbook the geometry of real per-

[12] Helmholtz, "Tatsachen, welche der Geometrie zu Grunde liegen." Fortschritte, 1868, Handbuch der Physiologischen Optik, 1866.

[13] John Stuart Mill, Gesammelte Werke, deutsch von Gomparz, 1869.

Hjelmslev, "Die natürliche Geometrie." Abhandlungen aus dem Mathem. Seminar der Hamburger Universität. 2. Band, 1923, p. 29.

[14] Moritz Pasch, "Vorlesungen über neuere Geometrie." B. G. Teubner, Leipzig, 1882.

[15] Springer, Berlin, 1928. Volume I is translated by E. R. Hedrick and C. A. Noble with the title: Felix Klein, "Elementary Mathematics from an Advanced Standpoint." New York, The Macmillan Company, 1932.

[16] A Set of Postulates for Plane Geometry Based on Scale and Protractor, Annals of Mathematics, p. 329 ff., 1932.

Birkhoff and Beatley, "A new Approach to Elementary Geometry." Yearbook of the National Council of Mathematics Teachers, 1929.

ception which I can show you after my lecture.[17]

Finally, we, in Germany, are of the opinion that we should tell the pupils the naked truth that mathematical laws in practical application can have only the nature of approximation.

Let us now recapitulate our ideas in the following statements:

1. We aim at the fusion of Euclidean and projective geometry in the spirit of perceptional applications.

2. The development of geometry takes place along the lines of representation, transformation and relationship.

3. Our school mathematics, from its earliest stages is empirical in nature. Empirical properties cannot, however, be logically defined. They can be perceptionally described.

4. Consequently we are not interested in a system of axioms. We attach no importance to the fact whether the needed statements and laws are logically dependent or independent.

5. The laws of geometry are considered in the sense of the relations between theoretical and practical physics, as mere approximations.

6. In the geometry of reality there are only lines of finite length, consequently there cannot arise in it a parallel axiom. Nothing is said about the intersection of two parallel straight lines at infinity—but it is clearly stated: Parallel straight lines never intersect.

7. We make no more distinction between a demonstrative geometry for beginners and other geometries, except that the pupils are gradually lead from the perceptual geometry to a more rigorous one.

8. When the pupils reach a mental maturity, that is, at about the age of 17–18, they are given the taste of the logical system of rigorous geometric thinking.

Not only in Germany, but also in America the movement "away from Euclid" made definite progress. The new book by Smith-Reeve-Morss, *Text and Tests in Plane Geometry* gives evidence of this fact. Besides this the developments in other countries are varied.

It was my purpose to present the main trends of the development of the inductive and deductive, the logically rigorous and empirical geometry and to mention the more important literature. Along this path of perception and thinking lies the development of school geometry for all of us even though we may go by different directions.

[17] (a) J. Hjelmslev. "Elementaer Geometri." 3 ed. Jul. Gjellerup. Copenhagen, 1931. (b) ———, "Die natürliche Geometrie," Abhandlungen des. Math. Seminars der Hamburger Universität 2 Band, 1923. (c) "Die Geometrie der Wirklichkeit," Acta Math. Bd. 40, 1915, p. 35–66.

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Mathematics in the Modern Curriculum for Secondary Education

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Two opposing tendencies are normally operative throughout the life of a social institution, the tendency to stabilize and the tendency to change. In a static society, the tendency to change meets successful opposition; in a dynamic society, the tendency of any institution to stabilize becomes the target of criticism and attack. Twice in the evolution of American secondary education, the dominant institution so successfully resisted change that it was displaced by a new institution designed to better meet the needs of a changed society. Today, public secondary education is again in a period of transition. The advocates of stability and change face each other as traditionalists and progressives.

The future of American secondary education depends upon the ability of the secondary school to adjust itself to the demands of changing social and economic life. In the brief period of the American public high school, we have already witnessed the outcome of the operation of these forces for stability and change. The junior high school, concerned with the education of children rather than the teaching of subject matter has made tremendous growth as an intermediate unit which is rapidly absorbing the first grade of the conventional senior high school organization. At its upper border the junior college with its concern about general education is showing rapid development. Between these two new units of educational organization the conventional senior high school still stands with its concern about subjects and courses and the mechanical organization of the curriculum. Some areas of the curriculum for secondary education have recognized the need for adjustment to modern life. Other areas have very largely resisted this trend for modernization. Mathematics would be classed with the latter group and mathematics teachers would in general be aligned with the traditionalists in opposition to the progressive movement. This is particularly true in the senior high school where no notable adjustment has been made in the curriculum within the present century.

What is the place of mathematics in the modern curriculum for secondary education? The answer to this question obviously depends upon the point of view of the person giving the answer. It should not be assumed that there is a single point of view representing the attitude of those who would identify themselves with the movement for progressive education. The point of view presented in this statement is justifiable only to the extent that it is consistent with a defensible philosophy of secondary education and in accordance with sound principles of education.

In the presentation of this point of view, facts or assumptions will be stated with only such elaboration as may be essential. It is impossible, within the limits of a brief statement, to elaborate as fully as might be desirable or to cite all supporting evidence. On the basis of the facts and assumptions advanced, a concluding statement will present a proposed plan of action which will indicate one concept of the place of mathematics in the modern curriculum for secondary education consistent with the facts and assumptions advanced.

Mathematics in the secondary school is enrolling a constantly decreasing proportion

of the student population. In 1910, 57% of all the pupils in the public secondary schools were enrolled in algebra and 31% were enrolled in geometry; by 1928 these percentages had decreased to 35 and 20 respectively.[1] One conclusion is very evident in the study of enrollment trends—mathematics is following the same path as the foreign languages, some of which have achieved extinction and others of which are rapidly following.

No teacher or school subject has a right which supersedes the best educational interests of the student. While this statement would probably not be openly denied by any teacher, probably few attempts have been made to reorganize the secondary school curriculum which have not come in conflict with the assumed vested rights of teachers. It is reasonable to expect that an individual who has spent years of his life in preparation for teaching in a certain field should resist attempts to modify the program for which he has directed his preparation.

The ability of the individual should be a basic factor in determining the materials and methods to be utilized in his curriculum. Although the science of education has demonstrated wide variation in the abilities of students, this information has not been adequately recognized. Although one evidence of this recognition is in the rapid decline of algebra and geometry as required courses, the inclusion of these courses in the elective offering, however, is restricted chiefly to the larger schools. The algebra requirement in the ninth grade has also been modified by the tendency in the junior high schools in the development of a continuous three-year course in general mathematics. This, however, has not affected a great proportion of the ninth grade population.

Perhaps one of the most striking pieces of evidence in support of the idea that all students should not be required to take algebra, is to be found in the study by Lindquist[2] based on the fifth annual Iowa Every-Pupil Achievement Testing Program. A comprehensive examination in elementary algebra was administered to over 9,000 ninth grade pupils just completing a year's course in that subject in 230 Iowa high schools. The first conclusion drawn by Mr. Lindquist from the data was that "a significant proportion of high school pupils are by reason of mental ability, previous training, and present motivation incapable of deriving enough value from ninth grade algebra as it is now generally taught to justify its being required of all students." As a specific illustration of the accomplishment demonstrated by these more than 9,000 students near the completion of a year's work in algebra, 33% could not write the formula for the perimeter of a rectangle with sides a and b; 30% were unable to write a formula for the area of the same rectangle; 95% could not solve the problem, "A dealer sold a suit for $42, making a profit of 20% of the cost. How many dollars profit did he make?"

It would be difficult to arrive at any conclusion other than the one stated by Lindquist unless we would want to charge all of this ineffective learning to poor methods in teaching. Certainly such contention would be difficult to defend. What was pointed out by Lindquist probably could be demonstrated with about the same results in a study of the achievement of students in geometry. Unpublished studies by Kendel at Colorado State College of Education have demonstrated repeatedly that the majority of several groups of elementary school teachers can not demonstrate ability in arithmetic on the Compass Survey Test beyond the seventh grade level.

Ability to achieve in mathematics increases with mental maturity. Ability to be successful in the study of mathematics

[1] Judd, Charles H. Chapter VII, "Education," p. 331 in *Recent Social Trends.*

[2] Lindquist, E. F. "The Gap Between Promise and Fulfilment in Ninth-Grade Algebra." *The School Review,* Vol. XLII, No. 10, pp. 762-771 (December, 1934).

depends in a large measure upon the ability of the student to think in terms of symbols and abstractions. The relationship between success in mathematics and the mental age of the student is high. On this basis it would be reasonable to postpone instruction in mathematics to the highest grade placement consistent with the job to be done and the needs for mathematical skills in the other curricular experiences of the student. One of the crimes of curriculum organization has been the consistent attempt on the part of subject matter specialists to push materials once established in the school curriculum down to constantly lower levels. As a result, students in the elementary school are working with problems in the development of skills which they cannot by any flight of imagination utilize in their experiences for many years to come, if ever. An excellent illustration of this tendency is found in a recent statement in *The Mathematics Teacher:* "First, the material in concrete geometry taught at present in grades 7 and 8 should be transferred to grades 5 and 6. Most of this material is too easy to be interesting and challenging to grades 7 and 8; and it is easy enough for grades 5 and 6. Second, there should be developed a new course in geometry for the junior high school."[3]

Wilson,[4] in this same journal, states on the basis of research which he cites that "Research shows that small children learn needed numbers as well or better without drill. Older children need some multiplication as well as counting, addition, and subtraction. If left without systematic teaching through grade 5 they will learn counting, addition, subtraction, and multiplication almost as well as if taught in the traditional school. Benezet's[5] experiment in the teaching of mathematics presents evidence that "children (at the close of the sixth year) who had no early drill on combinations, tables, and that sort of thing, had been able, in one year (the sixth grade), to attain the level of accomplishment which the traditionally taught children had reached after three and one-half years of arithmetic drill."

There is little scientific justification for the grade-placement of mathematics in the conventional program of studies. It is interesting that algebra and geometry with rare exceptions are placed in grades 9 and 10 of the conventional high school program. Why algebra should be in the ninth grade and geometry in the tenth grade is an interesting question for speculation. Apparently the justification for this placement goes back to the time when high schools commonly taught four years of mathematics because four units of mathematics were required for college entrance. In order to make possible the four-year program, the first course had to be placed in the ninth grade. Today schools offering four years in mathematics, or even three years in the senior high school, are a relatively small proportion of the total number of high schools. Yet we have maintained the conventional placement, although there is evidence to show that students in the eleventh grade who take first year algebra understand it more readily and achieve much more rapidly. The statement quoted above recommending the thrusting downward of concrete geometry from the seventh and eighth grades to the fifth and sixth grades is illustrative of one basis for placement of courses. Too little attempt is made to determine placement in terms of the contribution to the education of the child. Too little research is engaged in for the purpose of determining the age of most effective learning, and the application of research findings is too often resisted.

[3] Beatley, Ralph. "Third Report of the Committee on Geometry." *The Mathematics Teacher*, Vol. XXVIII, No. 6, pp. 329–379 (October, 1935).

[4] Wilson, G. M. "What Research Reveals on Proper Drill Content of Elementary Arithmetic." *The Mathematics Teacher*, Vol. XXVIII, No. 8, pp. 477–481 (December, 1935).

[5] Benezet, L. P. "The Story of an Experiment." *The Journal of the National Education Association*, Vol. 24, No. 8, pp. 241–244 (November, 1935).

Things once learned are soon forgotten unless they are used. This statement is so generally accepted that the citation of supporting evidence is not necessary. Yet we fail to recognize it in practice. Starting in the elementary school, things are taught although admittedly they cannot be used until sometime in the future. In ninth grade algebra the greater share of the year is generally devoted to the study of the equation. The student also learns about the formula and how to solve problems involving the formula. If we examine the conventional high school curriculum, we will find that the student probably will not have an opportunity to utilize his knowledge of the equation and the formula until he reaches chemistry or physics in the eleventh grade. An interval of two years has elapsed. What happens is generally recognized, however, our practice is not consistent with our knowledge of how learning takes place. Certainly any attempt to move algebra to a placement more immediate to the level in which its contributions are actually used would meet with opposition.

Learning activity is more effective if it is purposive. This statement, like the one above, is so generally recognized in theory that there is little need for elaboration. While we may concede the fundamental truth of this statement, we find it being violated in practice. Particularly in the field of mathematics is the violation extreme. As evidence of this, the simple question, "Where is the problem of motivation most difficult in high school teaching," will probably give the answer. Motivation and the teaching of mathematics are so commonly linked together that there can be but one conclusion, that mathematics in itself does not provide sufficient stimulation to insure effective learning activity on the part of the student. To insure this desired activity, the problem of motivation is one of the primary problems in the methods in teaching mathematics. There is no intention in this statement to endorse the extreme point of view that the child should do nothing that he does not have an original desire to do. Certainly the schools exist for the performance of a social function. What is to be learned cannot be left for the decision of the immature members of society. On the other hand, if any area of learning activity can be promoted only on the basis of high pressure, extrinsic motivation, it would be reasonable to assume that the situation would warrant careful examination for the purpose of determining the actual values of learning outcomes in the lives of the learners.

The outcomes of instruction in mathematics are primarily tool or service skills which are valuable only to the extent to which they are used. This statement is perhaps the first of a controversial nature in this series. It means essentially that mathematics must be defended primarily on the basis of functional values. There is little question that if there is no demand for a tool there is little justification for its possession. Wilson[6] says that "Adult usage is 90% covered by the four fundamental processes. . . . Most adult usage of the fundamentals is in simple form. . . . Beyond the fundamentals there are the processes of percentage and interest. These are not child processes. They are not much used by adults."

The cultural values of mathematics claimed by mathematicians have not been clearly identified. In spite of insistence of mathematicians on the justification of mathematics in terms of its cultural contributions, statements of cultural values of mathematics on the public schools level are rare and generally difficult to understand. Potter[7] identifies these cultural values as the ability to solve mathematical puzzles, interests in unidentified mathematical hobbies, acquaintance with such

[6] Wilson, G. M. "What Research Reveals on Proper Drill Content of Elementary Arithmetic." *The Mathematics Teacher*, Vol. XXVIII, No. 8, pp. 477–481 (December, 1935).

[7] Potter, Mary A. "The Human Side of Mathematics." *The Mathematics Teacher*, Vol. XXIX, No. 3, pp. 123–238 (March, 1936).

terms as "unequal beauty," "undivided attention," "a square deal," and others which "have implications and distinctions . . . seldom fully understood unless algebra is mastered," and an appreciation of form and design. The ability to read graphic materials, sometimes classed as a cultural value, is, of course, an essential tool skill and would be defended on the basis of a fundamental learning rather than as a purely cultural outcome. What are the cultural values of mathematics in the high school curriculum? Evidence of a lack of serious concern about the so-called cultural values in spite of the claims of their importance is to be found in the fact that tests and examinations in mathematics rarely include items designed to measure such outcomes.

The disciplinary values resulting from the study of mathematics do not transfer automatically to other areas of experience. Mathematics, like the foreign languages, have based one major justification on the assumption that the study of mathematics does something to the individual which remains even though the mathematics involved may be forgotten. Whether this concept embraces the idea that the result is keener-wittedness, the achievement of the scientific method of thinking or a method of proof, the fact remains that research has not demonstrated that this transfer is automatic or inevitable. Fawcett,[8] in a recent statement on teaching for transfer, states at the outset that "Out of the confusion of viewpoints relative to the transfer of training there seems to be one general agreement which is clear and well defined: transfer is not automatic." The Third Report of the Committee on Geometry of the National Council of Teachers of Mathematics,[9] continues on this same subject, "there is almost unanimous agreement that demonstrative geometry can be so taught that it will develop the power to reason logically, more readily than other school subjects, and that the degree of transfer of this logical training to situations outside geometry is a fair measure of the efficacy of the instruction. However great the partisan bias in this expression of opinion, the question, 'Do teachers of geometry ordinarily teach in such a way as to secure the transfer of those methods, attitudes and appreciations which are commonly said to be most easily transferable?' elicits an almost unanimous but sorrowful "No." Obviously, the attempted justification of geometry, and probably to a like degree, algebra, on the basis of disciplinary value, is an expression of hope rather than a statement of reality, an assumption of possibility rather than a declaration of achievement.

In so far as algebra and geometry may be defended on the basis of disciplinary values, unless these values are accompanied by functional, utility or cultural values it would be logical to take the position that if these values may be achieved by the uses of the functional materials it would be consistent with economical learning to disregard the disciplinary non-functional in the interest of expanding the disciplinary and functional experiences of the student. The question more specifically is, If the assumed disciplinary values may be achieved by the use of science learning materials, why would it not be more desirable to expand the science program and reduce the attention given to algebra and geometry?

The basic concept underlying the organization of a curriculum for modern education should be the development of skills, habits, knowledge, attitudes, interests, and other abilities which will function in the life of the individual. The history of education reveals three other concepts, the concepts of preparation, discipline, and the acquiring of knowledge and information. Long after school subjects have ceased to be re-

[8] Fawcett, Harold P. "Teaching for Transfer." *The Mathematics Teacher*, Vol. XXVIII, No. 8, pp. 465–472 (December, 1935).

[9] Beatley, Ralph, "Third Report of the Committee on Geometry." *The Mathematics Teacher*, Vol. XXVIII, No. 6, pp. 329–379 (October, 1935).

lated to the lives of the learner, they have been perpetuated in the school program on the basis of one of these concepts. The concept of mental discipline has already been discussed, as well as the concept of culture or the acquiring of knowledge and information for the sake of the knowledge and information acquired. It would be well for mathematics teachers in the high school to acknowledge that one of the first justifications for the existence of the mathematics program in the senior high school is that the colleges demand so many units of credit in algebra and geometry for admission and that their function is to enable students to meet these requirements. While there is a reasonable justification for questioning the justification of college entrance requirements of this type, the defense of the requirement is not the responsibility of the high school teacher of mathematics.

There is a large field of mathematical information and skill which is not recognized by the mathematics programs in the secondary schools. Practically all individuals some day buy insurance—life, fire, theft, automobile, health or accident insurance. What attempt is made in the high schools to teach students how to buy insurance intelligently? Taxation is a problem which touches every individual, not at some future date but right now. How many mathematics teachers teach the mathematics involved in taxation? Consumer buying is an area in which everyone is involved, yet where is secondary education concerned with this problem? Daily we are confronted by advertisements encouraging purchases on a six per cent deferred payment basis. How many students or adults can actually tell what the per cent of interest really is? This list could be continued—building and loan associations, home loan corporations, industrial loan banks and the statistics and graphical materials in newspapers are illustrative of other everyday experiences. Perhaps these are social problems responsibility for the teaching of which might be assigned to the teacher in the social studies. Unfortunately, the specialized preparation of the social studies teacher has not equipped him with the ability to teach the mathematics involved in these problems. In the junior high school levels, mathematics teachers have not hesitated to move into the social-commercial-vocational-arts field to secure problems of a more vital nature. Why shouldn't mathematics in the senior high school be concerned with these vital problems in everyday life? If only one job could be done, would it not be better to do this job even though it could be attempted only at the cost of the elimination of algebra and geometry from the high school program?

A Proposed Program in Mathematics for Secondary Education

On the basis of these facts and assumptions and the points of view presented in the foregoing statements, the following suggestions are made with regard to a program of mathematics in the modern curriculum for secondary education.

In the first place, the mathematics now taught in the elementary school should be subjected to critical study. Research points out definitely that much of the mathematics now taught in the elementary school is placed too low. Probably some of the mathematics now taught in the elementary school should be moved upward into the junior high school, and certainly much of the elementary school mathematics might well be eliminated.

Second, mathematics instruction in the general program for secondary education in so far as formal class organization is concerned should be extended only to the point that all students have a command of the fundamental abilities. If students promoted from the elementary school are deficient with respect to these fundamental skills, formal course work should be continued in the junior high school until such time as the deficiencies have been eliminated.

Third, algebra and geometry and the

more advanced courses which are sometimes included in the program of the senior high schools should be moved upward as far as possible. If only two years of mathematics are included in the high school program, these courses should be made available on the eleventh and twelfth grade levels.

Fourth, all mathematics other than the functional mathematics implied by the term, "fundamental abilities," and the mathematics necessary to intelligent understanding of social and personal problems should be made elective. In other words, algebra and geometry should be a part of the elective offering rather than a part of the required curriculum. Admission to these courses should be on the basis of the student's interest or his desire to meet college entrance requirements.

Fifth, mathematics instruction should be incorporated into the curriculum wherever it is needed. The mathematics teacher instead of having a class reporting to her daily to study problems in mathematics would, in cooperation with the other teachers, teach the mathematics involved in the other areas of experience at such time as the mathematical abilities would need to be taught. Specifically, instead of teaching the equation in ninth grade algebra the mathematics teacher would work with the physics or chemistry teacher, if that is where the equation was first used, and would there teach the equation in the situation where its use is needed.

Many significant problems are involved in the application of this proposed program. What will be the attitude of other teachers whose class procedures and time allotment will be interfered with if mathematics is taught in connection with such activities? Will all the mathematics which needs to be taught as a part of general education be adequately provided for in a flexible program of this type? If practice is not formally continued through the junior high school grades, will not the fundamental skills be forgotten?

For three years the Secondary School at Colorado State College of Education has been working into the program which has been proposed. The experience of these three years has demonstrated that mathematics, instead of being eliminated from the curriculum, is serving more effectively and economically an essential function in the education of boys and girls.

Hails Age of Mathematics*

A new name for the present age has been suggested by the mathematicians. It is the Age of Mathematics.

In some ways it is a good name. For every science, in the last analysis, is based upon mathematics. The engineer, the chemist, the physicist, the astronomer, all must use mathematics. Even the biological sciences are growing mathematical.

Physiologists, concerned with the effects of the powerful hormones secreted by the ductless glands, the electric currents which flow along nerves, the electric waves in the heart and the brain, the problems of allergies, and others, find themselves entering the realm of biochemistry. Here they must employ that modern branch of chemistry known as physical chemistry, a branch which is notable for its continuous demands upon mathematical theories.

The only trouble with the name, the Age of Mathematics, is that every age has been an age of mathematics in the sense that it progressed as far as its mathematics would permit.

It is important to distinguish between mathematics and calculations. The mathematician is concerned with the creation of mathematical theories and mathematical methods. Once these have been devised, their application can be turned over to a machine.

Addition, for example, can be accomplished more rapidly upon an adding machine than by hand. It is possible today to add.

Machines can be had today to add, subtract, multiply, divide, extract square roots and perform other routine calculations. Even more difficult types of calculations can be performed with the aid of machines.

It was entirely appropriate, therefore, that the first two days of the recent Harvard Tercentenary Celebration should have been given over to mathematicians. Men of affairs who pride themselves on being what they term "practical" may ask the use of a new arithmetic invented by Dr. Carnap or the new geometry of Professor Cartan. The answer is that time alone will tell. No doubt there were "practical" men who sneered at the calculus of Newton and Leibnitz.

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The Social-Civic Contributions of Business Mathematics

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It has been recognized for some time that students who are being educated for business require some special training in mathematics. Commercial arithmetic was among the first of the so-called practical subjects to be introduced into our secondary schools. It still remains one of the most important of the commercial subjects, in terms of enrollment, in the public and private secondary schools. If it has not made the contributions to the total education of the pupil that might have been expected of it, it is because, in common with other secondary school subjects it has suffered from a failure to define the major purposes of the course in the proper relationship to the aims of all education. When we consider the function of mathematics in the commercial curriculum, our questions must be somewhat as follows:

What are the purposes of education, and in particular, of commercial education?

To what degree can these purposes be achieved through the teaching of mathematics?

How are the aims, content, and teaching procedures in mathematics to be selected so as to accomplish these aims most effectively? It is our purpose briefly to examine these questions.

It is interesting to note, as one examines the statements as to the purposes of public education, that there is an increasing emphasis on the social-civic aim—that of preparing the individual to take his place as a cooperating member of society. There is a growing insistence on a curriculum that will explain the nature of the social and physical environment, and give the citizen the means to solve the problems that arise from it. One of the important purposes for which society is organized is that of providing the consumer with commodities and services. The organization of society for this purpose may be termed the economic system, or business. An understanding of the economic system, along with other aspects of the social organization, is to be expected of each pupil that has passed through the secondary school.

For the individual that is to take an active part in production, whether as an entrepreneur, a clerk, or an executive, a more detailed understanding of the purpose and nature of the economic system is required. It is the function of the commercial curriculum to provide for the needs of this group. The individual subject-matter fields are, of course, the means by which the purposes of education are to be accomplished. It follows that the primary aim of any given field must be to provide the pupil with:

1. An understanding of the nature and purpose of the economic system, together with the skills and information that are the prerequisites to such an understanding.
2. An insight into the social principles that are basic in our organization.
3. Those attitudes and ideals that are required if these principles are to become effective.

If business mathematics is to fit into this picture it should be a means of explaining the nature of the business world, and of equipping the individual with the means of solving business problems. Unless we place more confidence in mental discipline than we pretend. the course in business mathematics should consist of a mathematical study of economic problems and topics. The basic aim is not skill in the manipulative processes, although it is not necessary to underestimate their importance. It is rather the habit of mathematical thinking, and the ability to analyze a problem logically and quantitatively. It is when it is taught from this point of view that mathematics can best

achieve the social-civic values of which it is capable.

The nature of the course when organized along these lines may best be understood if we consider an outline of an economic topic, illustrating how a mathematical approach may be utilized to develop the essential concepts, attitudes, and principles. For this purpose we may consider a teaching unit on *The Consumer*.

There can be little question as to the need for including in the course a topic on the consumer. In the first place an understanding of the economic system and its purpose is impossible without a concept of the consumer as a motivating force, directing and financing the activities of industry. In the second place, there is a clear need for a shift in the emphasis in our teaching: a shift from an emphasis on production to an emphasis on distribution. In considering the problems of the consumer we are forced to give some consideration to the distribution of our national income, and the division of the products of the system among our population. In so doing we may be led to the development of a technique for teaching the problems of distribution as effectively as we have taught those of production in the past.

There are at present several topics in the majority of courses in business mathematics that might be classed as consumer education. This material, however, may be criticised as unsuitable for the outcomes desired on three counts: (1) It is not organized to develop the concept of the consumer as an economic entity; (2) there are no centralizing social or economic principles developed; and (3) the teaching is not planned to develop the attitudes that are an essential part of the outcomes. The effective organization of material with a view to achieving the social-civic outcomes involves three steps:

1. *The selection of the general social principles that are to be brought out, together with the attitudes and ideals that are necessary if the principles are to be effective in conditioning the behavior of the individual.* The social principle to be developed in a unit on the consumer may be a matter of individual preference on the part of the teacher, but a typical principle might be this: Each individual is under obligations to himself and to society to be an efficient consumer. The attitudes that will cause the individual to act on this principle are difficult to define, but they will be such as to arouse annoyance at violations of the principle, on the part of himself or others, and at any conditions that prevent his own achievement of efficiency.

2. *The selection of topics that are most suitable to bring out these principles and attitudes through a mathematical approach.* The topics that have proved suitable in a unit on *The Consumer* are five in number—Financial Management of the Home, Rent, Operating Expenses, Purchasing Activities, and Savings and Insurance.

3. *The development of each topic in such a way as to achieve the important outcomes of the unit—the acquisition by the pupil of the principles and attitudes.* Each of the topics just mentioned is developed through a mathematical study so as to bring about an understanding of the consumer as a part of the economic system. In studying each topic the pupil must come to an understanding of the favorable or adverse effect of his actions as a consumer, not only on his individual interests, but on the economic system as well. We may examine each topic briefly to see how the material is selected and organized for this purpose.

Financial Management of the Home

The analogy between the financial problems of the family and those of a business concern becomes clear when the cash account and the budget are utilized as devices for the intelligent control of expenditures. Besides its use in studying the techniques for effective financial management, however, the mathematical approach is valuable, even at the ninth grade level, in analyzing the economic and social

significance of family income. One may for example, make a quantitative study of such problems as the distribution of the national income among the families in this country, and the relationship between income and standards of living. There is available, for this purpose, a considerable amount of valuable material on the proportion of families receiving various incomes, and the utilization of income at each level.

Two types of exercises have proved effective for this purpose. One is the interpretation and construction of graphs. The excellent and attractive graphs in the recent report of the Brookings Institution[1] for example, furnish excellent material for practice in reading graphs made from such data, and the tables in the report, as well as the budget itself, lend themselves very effectively to the graphical presentation.

The second type of exercise is the use of percentage calculations in distributing various incomes among the budget items. These calculations are useful in developing an understanding of the variations of standards of living with changes in the income level, such as is expressed in Engel's Law. By considering the adjustments a given family must make as it adapts itself to successively lower income levels, as has been a common necessity during the past few years, it is clearly seen how the *proportion* of the income spent on food, clothing, and shelter increase, while the *absolute amount* available for each purpose decreases. Similar studies of families with rising incomes are equally valuable.

Rent

While the topic of rent is of great economic importance, its implications are little understood either by the consumer or by many business men. A study of rent includes an analysis of the expense of owning or renting a house, together with some of the problems relating to each item of expense. Two interesting questions that may be considered are: What items are included in rent if one owns his own home? and, how much may a family with a given income plan to pay if it decides to purchase a home?

The first of these questions is, of course, fundamental to such commonly considered problems as these:

How much rent should one pay for a given house?

Is it cheaper, in a specific community, to own or rent?

What community factors may act to increase or decrease the rent for a given house?

For a quantitative study of these questions it is, of course, desirable to have the data for one's own community. If these cannot be obtained, then the next best thing is to use the figures that are typical for the country. Usually the cost per year of owning a house, in terms of per cent of its value, is as follows:

Taxes	2.1%	Insurance	0.2%
Maintenance	1.4%	Interest	6.0%
Depreciation	2.5%		
		Total	12.2%

After a series of shorter exercises intended to bring out the significance of each item, one may propose a problem similar to this:

> Mr. White is living in a house owned by Mr. Black. He is paying $65 a month rent, and believes it should be reduced. The house is valued at $7,500, and is assessed for 50% of its value. The tax rate is 50 mills per dollar. It is insured for 70% of its value for fire, at a rate of 24¢ per $100 annually. Maintenance and depreciation average about $250 per year. Mr. Black should receive 5% per year on his investment in the house. Calculate what it costs Mr. Black per year to own the house, and tell whether you think Mr. White is paying more rent than he should.

Interesting developments from this problem are secured if one considers the effect on the cost of owning the house, and probably on the rent, of such factors as these:

[1] M. Leven, H. Moulton, and C. Warburton, *America's Capacity to Consume*. Brookings Institution, Washington, D. C. 1934.

a. The city issues bonds, and the tax rate is increased 10 mills per dollar for debt service.

b. The fire department is inefficient and insurance rates are doubled.

Other factors that will affect items of expense will occur to the reader.

The second important problem, that of determining how much a family with a given income might plan to pay for a house, is, of course, an approximation which, like all calculations depending on the budget, must be adapted to the community. We may consider this problem:

Mr. Brown has a wife and two children. His income is $150 a month. He is considering the purchase of a house. Assuming that his income will be the same for some years, what would you recommend that he plan to pay for a house?

According to the budget used by the writer, a family of four, on an income of $1,800 per year, may pay 24% of it for rent. The steps in the solution, then, will be:

1. Mr. Brown may spend 24% of $1,800, or $432, for this item each year.

2. According to the figures given in the problem above, the annual cost of owning a house is 12.2% of its value.

3. $432 is therefore 12.2% of the cost of the house he should consider, which is $3,541.

The value of such an estimate, as well as its approximate nature, should be understood by the pupil.

An interesting variation of this problem occurs when Mr. Brown considers purchasing the house through a loan from a building and loan association, and tries to make the monthly payments fit into his budget, along with the other expenses of owning a house.

Operating Expenses

A study of the operating expenses in the home includes an analysis of items such as fuel, light, telephone, and so on that are of most interest and importance. As commonly taught, most of this material comes under the head of Worthy Home Membership, among the Cardinal Principle. The degree of which, in studying the cost of electric refrigeration, for example, one might go on to a study of public utilities from the point of view of the consumer, and include the cost and economic significance of the T.V.A., the Boulder Dam, and Fort Peck projects, and so on, is a question that deserves further consideration.

Purchasing Activities

The problems of purchasing for the home are similar in many ways to those of purchasing and buying in business establishments. The topic may be motivated by a statement of the weekly expenditures of the American public on food, clothing, and luxuries, which ranges from a half to three-quarters of a billion dollars per week. Supposing that various percentages of this amount may be saved through careful purchasing, the determination of the amounts that might be added weekly to those available for education, recreation, or savings offers an interesting group of problems. This introduction leads directly to the question of the means for economical purchasing. These, in general, may be studied under five headings: Buying in bulk, on sales, for cash, in quantity, and the selection of the suitable quality. Each of these methods may be studied through practical problems selected from the advertising section of the daily paper by the pupils. When all five methods have been studied, a list of articles to be purchased may be prepared with regular prices, and the percentage saved by careful buying may be determined. Referring back to the $500,000,000 the saving of even a cent on a purchase becomes important in terms of percentage saved.

The study of the question of purchasing in packages as compared to purchasing in bulk offers exceptional opportunity to show how the demands of the consumer direct the activities of industry, whether intelligently or otherwise. There are, of

course, certain commodities that the consumer does not care to purchase in bulk. On the other hand, the packaging of goods has gone to extremes, and if the consumer does not exercise an influence in the proper direction a considerable portion of the resources of industry will be wasted in producing unnecessary packages. The increased cost to industry and to the consumer becomes apparent when the cost of the same commodity in bulk and in packages is compared in terms of per cent.

The fact that the consumer is deprived of information that is necessary for an intelligent choice among the end-products of the economic system is apparent when a study is made of buying the correct quality of food, clothing, or other materials. We know that the merchant can specify the grade or quality that he needs, and so compare prices and have redress if an inferior grade is substituted. To compare prices or protect himself from fraud the consumer, from whom the information about grades has been withheld, must personally inspect his purchases, and be qualified as a grader and sorter in dozens of different lines.

The question of buying for cash introduces the idea of how the cost of commodities may be increased by unbusinesslike behavior on the part of the consumer. Not all credit business is bad, of course. The consumer should, however, be aware of the cost to the merchant of open accounts, and of installment selling, which must be passed on to the consumer in advanced prices. The first of these can easily be shown by studying the increased amount of capital needed for a credit business, on which interest must be earned, as well as the cost of bad debts, which up to the present have not been sufficiently respectable to permit their introduction into the classroom. The study of the cost of installment buying requires a sound method of approximating the rate of interest on the unpaid balance, but the information that can be obtained from advertisements justifies the learning of the calculation.

Savings and Insurance

The topic of savings and insurance is the least satisfactory of the topics on the consumer, although it is the one on which we have had most experience. Perhaps these two facts are related. For years we have taught the nature and purpose of investments, the agencies available to the individual for this service, the nature of investment instruments, and the characteristics of a sound investment. We have passed on the propaganda of insurance and savings institutions uncritically and have practically sold insurance in the classroom. We have held up the idea of thrift as an adequate means of protection against economic disaster, and thereby left the impression with the pupil that anyone who unable to protect himself against economic hazards and dependency has a history of thriftlessness.

To understand what is attempted by way of individual protection in our present social order, it is, of course, desirable that the pupil consider some problems such as these in the higher income levels where the plan actually functions:

> Recall what you know about the nature and use of the savings bank, building and loan associations, and the various forms of insurance, and set up a plan for savings and protection for these families. Use the amounts provided in the budgets for each income.
>
> 1. Mr. Williams is 30 years old, and has an income of $6,000 a year. He has a wife and three children. He wishes to secure adequate protection for them in case of his death, and to have sufficient savings to retire when he is sixty years old.

It is probable that more ideas and attitudes will be developed in the discussion of the relative merits and defects of the various plans that are presented than would be the case if the problem required a single answer that could be scored right or wrong.

When we leave the higher income levels, however, we soon encounter a definite limit to the amounts available for protec-

tion. The report of the Brookings Institution gives us these figures:[2]

Income	Per Cent Saved
$4600 and over	38
3100 to 4600	17
2450 to 3100	13
2000 to 2450	11
1700 to 2000	8
1450 to 1700	5
1250 to 1450	3
950 to 1250	0

These figures are particularly significant when we consider that less than twenty per cent of the families in this country, even in so-called prosperous times, received over $4,500 per year.

We have only to follow through a series of problems in budgeting the income at the middle levels up through the lower eighty per cent of the income groups to encounter such questions as these:

What proportion of families can afford protection against the hazards of economic disaster? What per cent of men and women of a given age are certain to be indigent in old age by reason of subsistence and under-subsistence income during the earning period? Whose responsibility is the protection of these persons who are unable to protect themselves through personal savings and insurance?

A mathematical analysis of the problem from this point of view opens up the whole field of economic security. It suggests the possibility of a mathematical treatment of social questions, and the question, not of how far we can go, but how far we should go in educating for social progress.

In outlining the organization of the teaching material for *The Consumer* it has been our purpose to see how the subject-matter field can be utilized to bring out the social concepts. Thus mathematics is used to develop the ideas of financial management of the home, rent, and so on. They form the basis for the understanding of the principles and the development of the attitudes that are the outcomes of importance. In this recognition of relative values—the subordination of the subject-matter fields of the concepts, and the concepts to the principles, attitudes, and ideals—undoubtedly lies the key to the attainment of the social-civic objectives of education.

[2] Op. cit. from tables and graphs pages 95–6–7.

Fulfill the Trust of Our Fatherland

(A posthumous article by Dr. I. P. Pavloff)

What would I wish for the youth of my fatherland who devoted their lives to science?

First of all—thoroughness. I can never talk calmly about this most important condition for fertile scientific work.

Before you attempt to reach the pinacles of science study—its A B C. Never attack the second step before you master the preceding one. Never try to cover up insufficient knowledge even with the boldest guesses and hypotheses. No matter how attractive is the rainbow of a soap-bubble, the bubble must eventually explode, and you are left in confusion.

Train yourself to patience and self-control. Learn to do detailed work in science. Study and accumulate facts. The wing of a bird, however perfect it might be, cannot lift the bird in the air unless it strikes against this air. Facts are the air of a scientist; without them you cannot fly. Without them your theories are worthless.

But when you study, experiment, observe, do not remain on the surface of facts. Don't become curators or archives of facts. Try to penetrate into the secret of their origin. Doggedly seek the laws that govern them.

Secondly—humility. Never think that you know everything, and however highly appraised have the courage to say "I am ignorant."

Don't let conceit rule you. It will make you obstinate and will deprive you of helpful advice and friendly cooperation. It will make you lose the sense of objectivity.

In the collective directed by me the surrouding atmosphere is an important factor. We are engaged in one common work, and everyone moves it further according to his abilities and strength. We seldom distinguish between what is "mine" and what is "thine," but the common work benefits from this.

Thirdly—passion. Remember that science demands all his life from a man. Two lives would not suffice for that if you only had them. Science demands great passion and tremendous effort of a man. Be passionate in your work and in your researches.

Experimental Studies of Large Unit and Individualized Plans of Supervised Study of Secondary School Mathematics

By HARL R. DOUGLASS
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University of Minnesota

WHILE no one study may furnish the basis for definite or permanent conclusions with respect to the relative value of various proposed plans of directing the study of high school mathematics involving large unit assignments and individualized progress, one may better evaluate the claims made for and against such plans, if one is familiar with the results of those experimental studies which throw light on the question.

Ambitious or speculative authors have written not only articles but books devoted to the description and theoretical support of elaborate procedures of this sort, developing systems of steps and terminology. These schemes have been adopted widely and somewhat uncritically by gullible classroom teachers. The plans may prove to be contributions to methods and again they may prove to be but additions or alterations or fads. At least one should examine as quickly as it is available the experimental data relative to these innovations.

The author of this report has examined nearly all the reliable published studies in this field and in addition a few unpublished graduate theses. They do not warrant any indisputable conclusions as to the superior effectiveness of large units as compared to daily assignments or as to individualized versus group progress plans, though when all the reported experiments are considered together, the tentative presumption should probably be in favor of the individualized larger units method if intelligently and skillfully taught. This appears particularly the case when pupils of superior ability are considered. Brief abstracts of the available studies follow:

Williams (1931) matched two sections in advanced algebra composed of average and superior students, one taught according to a modified Winnetka contract plan and the other taught by the conventional daily regulation plan, all taught by the same instructor in Altoona (Pa.) High School. Pairing was done on the basis of I.Q. based on the National Intelligence Test and marks in beginning algebra. Gains were measured by the Thorndike Algebra Test for the first semester and the Co-Operative Algebra Test for the second semester, Form A of the respective tests being given at the beginning of the semester and Form B at the end. In seven of the eight comparisons of gains the differences favored the contract method. In the one exception the difference is not statistically reliable.

Stallard (1932) at the Longfellow Junior High School at Wauwautosa, Wisconsin, found that for superior students, the large unit plan of supervised study was superior to daily recite-study divided period plan of supervised study, and that the daily plan was better for pupils of average ability though the latter differences were not completely reliable.[1] With 24 pairs of pupils of higher I.Q (105 to 121, mean = 116) equated as to I.Q. (Otis Self-administering Intelligence Test) and scores on the arithmetic section of the Stanford Achievement Test, he found the differences in gains on the Douglass Standard Survey Algebra Tests during the first semester of beginning algebra favoring the large unit plan with a minimum of recitation. These differences were such as 97 times in 100 would not

[1] By complete reliability is meant that there is less than one chance in 100 that the difference obtained could have resulted from chance errors of sampling and measurement.

have occurred as the result of chance. With 24 pairs of pupils of average mental ability (I.Q. from 94 to 116, mean = 105) he found differences in favor of the daily recite-study divided period plan the probability that which were due to chance was 17 in 100.

Hunziker in the Teton County High School at Choteau, Montana (1933), on the contrary found the daily recitation plan superior to a large unit plan of supervised study during the second semester of both beginning algebra and geometry, though negligible results in favor of the large unit plan were obtained for the first semester of algebra. Equating his groups of 19 pairs of pupils in two section of algebra and 15 pairs in two sections of geometry, and reversing the methods with the groups for the second semester, he measured the gains in algebra by means of the Douglass Survey Algebra Tests and a test prepared by himself, and the gains in geometry by means of a test of his own making. In each semester in geometry the difference favored the daily recitation plan by an amount attributable to chance errors by a probability of only 14 in 100. In the second semester in algebra the difference favored the daily recitation plan, and was such as could be attributed to chance errors by a probability of less than 1 in 100. Neither plan seemed peculiarly suited to bright or less able pupils.

Gadske's findings (1933) based on 100 pupils in the Community High School at Carbondale, Illinois, unmistakably favor large unit assignments and supervised study as compared to a recite-study plan of supervised study in first year algebra. His groups were paired on Otis Group Intelligence Test I.Q. and on scores made on the arithmetic and reading sections of the Stanford Achievement Test, and progress was measured in terms of gains on the Columbia Research Bureau Algebra Test. The differences were easily statistically significant at the end of each semester, particularly for the second, and seemed to apply to pupils of all levels of ability.

Drake (1933) working with 22 pairs of average and superior pupils in first year algebra in the University of Minnesota High School compared two methods of employing a three-level large unit assignment plan—in one of which the pupils progressed through the course of their own individual rates and in the other the pupils started each subunit of approximately a week's work at the same time. The experimental groups were paired on age and the median of five I.Q.'s based on five different mental ability tests and achievement after the first five weeks of work. Progress during the experimental period was measured by means of the State Board examination in Algebra, two final examinations and unit tests. Differences in the State Board and final examination scores favoring the individual as compared to the individual group plan were statistically significant, the chance being less than 1 to 100 in each case that the difference could be attributed to chance errors, though the individual method required more student time. The superiority of the individual group method seems to apply equally to pupils of greater and to pupils of lesser ability.

Eilberg (1931) experimented with two advanced geometry sections from the South Philadelphia High School for Boys during each of three successive terms of five months each in an effort to measure the relative effectiveness of the Dalton Plan as compared to the conventional recitation method. The sections, composed of 30 boys each, were equalized by the use of the following determinants: (1) chronological ages; (2) I.Q.'s secured by the use of the Otis Group Intelligence Scale, Advanced Examination, Form A; (3) school grade; (4) junior high school training or not; (5) school course pursued by the student—academic, commercial, or mechanic arts; (6) past success in Algebra I, Algebra II, and Geometry I; (7) derived Geometry I scores. The

medians were higher for the Dalton Plan classes on 33 out of the 36 tests, given at intervals—twelve to each pair of sections. Of these differences, twelve were great enough to be statistically significant, that is, beyond the reasonable probability of being attributable to chance. Five others were not significant. At the end of each semester the Dalton classes made higher medians on the Columbia Research Test in geometry, though the difference for no one semester considered separately was statistically significant. Apparently the superiority of the Dalton Plan was greatest for the students of near average ability while for the upper and lower groups there was little difference.

Allen's study (1933) is unique in that she attempted to measure the relative effect of two methods—the individual and recitation methods—not only upon gains in achievement in subject matter but also upon personality, honesty, vocational interest and opinions on public matters. Two groups, 50 in all, of 11th grade students in plane geometry at Altoona (Pa.) High School were paired on scores made on Terman Group Test of Mental Ability, Cooperative Geometry Test, and the Rogers Test of Mathematical Ability (geometry section). In addition to progress in mathematical abilities, changes in scores on the Berneuter Personality Inventory, Miller Self-Marking test, (honesty) Strong Vocational Interest Blank and Watson Test of Public Opinion (prejudice), were studied. In the individual unit plan each unit was assigned a definite time limit. Superior gains of statistical reliability were found for the individual method with respect to the following personality character traits: dominance, submission, honesty, and lack of neurotic tendency, as well as in achievement in geometry.

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A midshipman not too good in his mathematics was told by his captain to take his bearings. He did so and brought his calculations to the captain's cabin. Glancing at them the captain exclaimed: "Young man, kneel down, and take off your hat—you are on holy ground. According to your observations we are in the middle of Westminster Abbey!

Modern Mathematics in Germany*

By PROFESSOR W. THRELFALL
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SINCE I am speaking to you about modern German mathematics, I wish to call your attention to a most important subject, namely to the world which surrounds us, and to our scientific knowledge of its extent in space and time. There can be no doubt about the fact that this world we are living in is not the best of all possible worlds. Financial, industrial, and political diseases, you know them just as well on the other side of the great pond as we do on this side. Nevertheless in one respect we are living just now in a golden age. The world of science is in an excellent state and few eras have seen as important successes of mathematics and physics as ours.

Ancient problems that have been puzzling mankind for thousands of years have been solved during the last fifty years. For instance the impossibility of squaring the circle has been proved by Lindemann.

But don't be afraid that I shall worry you with an enumeration of such things. I only want to show you a parallel between the great changes of scientific thought which have recently come about and those of the sixteenth century, grouped around the name of Copernicus.

The great discoveries of yonder times were the exploration of the oversea-countries by circumnavigating the globe, the telescope that showed the moons of another planet and so on. You will see some famous instruments of that period in the Zwinger Gallery here in Dresden. The result of this expansion of human experience was, as you well know, the heliocentric system. The theatre of this our world was no longer a great metropolitan opera house in the center of the universe, but had to be degraded into a small provincial theatre. No sane person could any longer doubt the finiteness of the terrestrial countries and the spherical form of our planet.

Now one might think that all the world has been thankful to the pioneers and scientists who revealed the secrets of the planetary system. However, quite the contrary is true. Luther, the political and religious revolutionary called Copernicus a fool and he rejected the heliocentric system. A delightful proof of this fact is to be found in the Zwinger Gallery, where, in the mathematical and physical section, there is a collection of ancient horoscopic clocks. Now it will be noticed that nearly all clocks dating from the time of the Lutheran Reformation, when coming from Lutheran countries are based on the old Ptolemaic system and those from Roman Catholic ones on the new Copernicanian system. Moreover the survivors of the Magellan expedition were shut up with water and bread in a cathedral, when they returned from the first circumnavigation of the globe, or rather some orthodox minded people wanted to deal with them in that way. For according to their logbook they came back with a date wrong by one day. Even in those days many people knew the reason why.[1] But notwithstanding the poor sailors who had sometimes lived on rotten water and mice on their ship and had fasted enough, were charged with having fasted on the wrong days.

It took a long time till the truth of the new ideas was allowed. There are two excuses for mankind. Firstly the scientific theories were mixed up with the political and religious fights of those times. Even men of science then dreaded the authorities, they lived in fear of their lives and

[1] For instance, Cardinal Contarini.

* An address delivered to a group of American Mathematics Teachers under the leadership of W. D. Reeve at the Technical High School in Dresden on August 2, 1935.

their families or they were anxious to fall behind the rising torrent of folly.

Besides, it seems to be, if not a definition, though a criterion for the importance of a scientific theory, that it is not shut up in the college halls of the universities, but excites the hearts and souls of men and must have martyrs amongst those who confess it.

The second excuse for the people of those times is that it was much more difficult for them, than we now may think, to get a grip of the new ideas. If they accepted the globosity of the earth, they had at the same time to resign to an absolute perpendicular direction, valid for all spots of the world, for its center and the antipodes. They objected to bringing their perpendicular plummet with them everywhere. They could not imagine a space without a privileged direction. A space where there is no *Below* and *Above* seemed nonsense to them, a remote philosophical idea without meaning. They could not afford the abstractive faculty—now so familiar to all of us—to live in a space without a gravitational field. They were at first shocked by the Renaissance view of the universe, given in Goethe's verses:

And swift, beyong conception's ranges
Earth's splendor whirls in circling flight;
A paradise of brightness changes
To awful, shuddering deeps of night.
The sea foams up, wide-spread and surging
Against the rocks' deep-sunken base,
And rock and sea sweep onward, merging
In rushing spheres' eternal race.

Only very clever and open-minded men clung to the new theories. But the most noble of them had been put in prison or gagged and silenced during the revolution of those times and to possess their papers or even to mention their names was dangerous.

Till now I have shown you the one side of the two parallels I wanted to draw before your eyes. We must keep in mind the two difficulties our ancestors had to overcome: The relativity of the perpendicular direction and the new and comparatively abstract opinion of the cosmic position of man, given by the heliocentric system.

Now I shall draw the other line. The great discoveries of our time are still more abstract. We have no more new continents to explore and newly discovered stars are not very exciting for us—I mean to say stars in the heavens and not those in Hollywood. But our scientists have got familiar with the unthinkably high velocity of light and electric waves, and they discovered the new world of the atoms. These discoveries are no less revolutionary for our lives than those of the sixteenth century. For on them are based the most important progresses of modern technics. There are two theories, both put forth in Germany, that led to strange new conceptions of the physical world: the theory of relativity and the quantum theory.

The theory of relativity closely corresponds to the Copernicanian revolution. Who once has understood the heliocentric system, never can return to the Ptolemaic system. And whoever has understood the theory of relativity never can return to the Newtonian system.

Even the difficulties to be met with are quite analogous. Firstly, to the impossibility of taking a plummet to every spot of the world and finding out a true perpendicular direction with it, there corresponds the impossibility of taking a watch to the farthest moving stars and ascertaining absolute synchronism with it. A physicist living on a star of some distant and quickly moving nebular system will not agree with us about the synchronism of certain events, just as we will not agree with our antipodes about the right perpendicular direction of an absolute space.

It may seem a paradox to deny the possibility of extending the time of our watch to moving bodies. But it is no more paradoxical to us than was the relativity of the perpendicular direction to the mediaeval scientists. And as they could not think of a space apart from a Below and Above,

modern people raise the objection that a space-time without absolute synchronism is a remote philosophical idea without meaning.

To a physicist familiar with the high velocity of light, this is not so. According to the restricted theory of relativity it is quite possible to extend by light signals the time from a gaugetower to all points on an immovable rigid body, to which the gaugetower is fixed. We have to presume that a watch is fixed to every point of this body. If then the signal, sent forth from the gaugetower at the time $t=0$, arrives at an observer at a distance d from the gaugetower, the observer's watch must be set, at the moment of the arrival, at the time $t=d/c$, c being the velocity of light. By this process all watches on the rigid body may be regulated by the clock of the gaugetower and synchronism may be defined on the rigid body in a way independent of the locality of the gaugetower. Now supposing that another body is moving with a constant velocity v (smaller than c) towards the immovable body, an observer on the moving body may gauge the watches of his body from a gaugetower fixed on it in the same manner as was done on the immovable body, the moving body also being supplied with watches at its different points. So far everything remains in the restricted theory of relativity as it was before. But should the clocks of both gaugetowers be set at the same time, say the time $t=0$, at the moment when passing each other, then the observer on the immovable gaugetower will observe, that the watches of the moving body passing him are slow. And the same observation will be made by the observer on the moving gaugetower with respect to the watches on the immovable body, to him appearing as moving. Taking into consideration the transformation-equations of the space-time of the two systems, the so-called immovable one and the so-called moving, one recognizes that there is no contradiction in that reciprocity. In particular there is no violation of the principle of homogeneity of space and time. For at all times there will be *one* point on the straight line joining the gauge-towers in both systems, where the two watches just passing each other will show the same time. One would have obtained the same gauging result, if one had at the particular point of time-coincidence exchanged each other's watches, instead of doing so at the moment of the passing of the gaugetowers. It must of course be understood that the actual point of coincidence varies in both systems. The two space-time-systems at one moment may therefore be represented on the two systems at another moment with conservation of all lengths and all times.

Of course a complete understanding of the theory of relativity cannot be obtained without the mathematical *formulas* for the space-time transformation. But we shall not go deeper into the difficulties to be met with in this theory.

Secondly, to the transition from the earth-disc to the moving earth-planet there corresponds a new cosmologic conception of the world, namely the transition from the infinite space to a closed space. When I speak of a closed space, I am making an expedition into an extra-scientific territory. For this consequence of the theory of relativity is not as well founded as is the theory itself. Notwithstanding it is extremely probable that our space, the space we are living in, is not infinite, though unlimited. We never will come to a dead stop, but we will return to our original place.

Take the case that I am looking straight forward from this place. If I am waiting long enough—that is about 50,000,000 years—I shall see flashing up before me the bald patch on the back of my head—provided I use a sufficiently powerful telescope. The universe is unthinkably large; as I mentioned just now, light would need, say 50,000,000 of years to go through the world and return. But there is no more Infinity. There is nothing left of the greatness of the world as a famous German poet (Schiller) described it:

Madly yearning to reach the dark
Kingdom of Night,

I boldly steer on with the speed of the
light;
All misty and drear
The dim Heavens appear,
While embryo systems and seas at their
source
Are whirling around the Sun-Wanderer's course.
When sudden a Pilgrim I see drawing
near
Along the lone path,—"Stay! What
seekest thou here?"
"My bark, tempest-tost,
Seeks the world's distant coast,
I sail tow'rd the land where the breeze
blows no more,
And Creation's last boundary stands on
the shore."
"Stay, thou sailest in vain! 'Tis Infinity
yonder!"—
" 'Tis Infinity, too, where thou, Pilgrim, wouldst wander!"

Nevertheless infinity still remains in the possibilities of explaining physical phenomena. It may sometimes appear as if truth had been discovered and the final system had been eventually evolved. As far as I am concerned I once was nearest to this conviction when Hermann Weyl succeeded in explaining the electromagnetic field as a geometrical characteristic of space-time. But still more recent ideas have overtaken even this seemingly final evolution of our conception of the universe.

To look at the closeness of the world from another side, imagine flat-fish living on the surface (or better: *in* the surface) of a large sphere and knowing nothing of a third dimension, as we know nothing of a fourth dimension. If these flat-fish went on the straightest way they could think of, they would describe a great circle of their sphere, as we admit that they know nothing of the surrounding three-dimensional space. It may be just the same with us, only one dimension higher.

But I shall not dwell any longer on the abstract speculation of a closed space. A matter-of-fact physicist will otherwise frown on me. I only wish to insist on one point: The idea of a closed world has put to the pure mathematician one of the most interesting and most important modern problems, that is to find a complete list of closed spaces for the physicist to select from.

If the world be closed, the physicist may discover and determine by however vague hypothesis its metrics, and he will decide on its special form. But the pure mathematician ought to be able to forecast all possible forms of closed spaces in the same sense that he can enumerate all regular polyhedrons and he is obliged to discover the possible relations between the metrics of a three-dimensional space or the four-dimensional space-time and its topological form in the large. This problem has been standing during the last few years in the center of mathematical research work. Especially at Princeton University in the United States and in Germany one is working hard on it, and the pure mathematicians are deeply enjoying through it "the heaven-sent faculty of reasoning, remote from the fuss of external Nature."

I am at the end of my lecture. Let me once again remind you of the parallel, I wanted to call your attention to: the relativity of the perpendicular direction and the revolving of the globe round the sun were the two important discoveries of the sixteenth century. Relativity of synchronism and possible closeness of the world are the two corresponding discoveries of our time.

All nations are now working on these problems hand in hand as much as they may fight with regard to other problems against one another. The leading thoughts of the two parallel scientific revolutions I mentioned, were put forth in Germany. As a born Englishman I may say it without boasting: What concerns philosophy the call has always been: Germans to the front!

ORDER THE ELEVENTH YEARBOOK NOW!

THE ART OF TEACHING

A NEW DEPARTMENT

The Mathematics Club at Curtis High School

By ESTHER SWEEDLER, *Staten Island, New York*

THE STUDENT studying physics, biology and chemistry can apply and test his text book theories in the laboratory but the mathematics pupil learns his rules and formulas without ever being given the opportunity to apply them. I recall with some amusement now my embarrassment when a youngster at a summer camp asked me how I would go about finding the distance across the camp's lake. I had just completed a major in college mathematics but the simple procedure necessary to find the distance across the lake was very vague in my mind.

I was not really impressed with the need for a mathematics laboratory until I taught trigonometry. Trigonometry—the subject of three-angle-measurement, the subject in which the student learns how to find heights and distances across inaccessible places on a piece of paper! No wonder some students persistently make the mistake of representing an angle of depression as measured from the vertical instead of the horizontal. Give that erring student a transit, let him look at it, let him feel it, let him manipulate it and he will no longer represent an angle of depression incorrectly.

The Mathematics Club at Curtis High School has tried to bridge the gap between text book study and actual application. Membership is open to students in trigonometry, solid geometry and advanced algebra. This semester the club boasted a membership of 36 boys. The work of the term is divided into two sections, the study of the slide rule during cold weather and the use of simple surveying instruments when the weather is favorable. The members learn how to apply their text book knowledge to field problems. The angle mirror, plane table and transit are studied. A forty minute period unfortunately limits the group to simple problems. The projects include a map of the walk leading to Curtis made with the angle mirror, a map of a section of the campus made with the plane table using the principles of radiation and intersection, and the finding of heights and distances with transit.

These lessons in the field need no teacher motivation, they motivate themselves. The enthusiasm of the boys is such that the ringing of the bell causes a grunt of disapproval. Some of the boys did independent work week ends, making maps of their front lawns and back yards. One Sunday morning was devoted to teaching the use of the level and target rod. The boys found the difference in elevation between Curtis High School and Borough Hall of Richmond to be 60 feet. They could hardly believe that in the seven minutes required to walk from the school to Borough Hall they were actually going down about four flights of an apartment building.

Instruments are expensive and the club has been fortunate in getting them. We can now boast of five angle mirrors, two plane tables, three transits, a hypsometer, a sextant and dozen range poles. Some of these were obtained through the generosity of the General Association and some through an unused book fund appropiation. Not all of our instruments are purchased. Three of the angle mirrors, an aldidade and two slide rules were made by math club boys. The range poles were made in the shop and then the Mathe-

matics Club had a good time painting the red and white bands. Mr. John Wiseman, a fellow teacher and unstinting helper, made a transit that measures vertical and horizontal angles accurate to the nearest degree. This simplified transit has the advantage of clearly showing the principle involved in the solution of a problem without the need of first learning how to manipulate it. Borough Hall has permanently loaned us a discarded city transit. Unfortunately the cross hairs which are obtained from a spider's web were broken. I wrote to the Keuffel and Esser firm but found that the process of putting two hairs from a spider's web into the telescope of a transit was an expensive proposition. I explained the dilemma to my husband, who is a civil engineer, and he willingly volunteered to undertake the job himself. Spiders' webs being scarce in apartment buildings, he finally decided to donate two of his own fastly thinning hairs to the transit. On this instrument the boys have learned how to read the fine vernier accurate to twenty seconds, how to level it and how to manipulate the fine adjustment screws. With this instrument they have done such problems as measuring the angles of a triangle, prolonging a straight line, measuring angles about the horizon and finding heights and distances. The third transit is a Starrett which we purchased through the Lafayette Instrument Company.

I am glad to be able to say that the Mathematics Club is more than a laboratory in which text book information is applied. It has given some of the boys an opportunity to use their abilities as leaders. During the study of the slide rule, boys who were in the club for more than one term helped the new members. Since I could not possibly supervise 36 students in the field, I divided them into two groups and turned one group entirely over to a boy by the name of Arthur Dean who had been in the Mathematics Club for three terms. He taught and supervised his group in the field with the greatest success. His picture appeared in the New York Sun of June 6th.

I recall my surveying professor at the University of Michigan asking me why in the world I, one of the weaker sex, wanted to learn surveying. When I replied, "So that I might teach it to my mathematics students," his response was, "Can't be done, can't teach it to high school students." But it can be done and we are doing it at Curtis High School. I hope the time will soon come when work with field instruments will be an integral part of the high school course in trigonometry approved by the Board of the Regents.

PLAYS

Back numbers of *The Mathematics Teacher* containing the following plays may be had from the office of *The Mathematics Teacher*, 525 West 120th Street, New York.

A Problem Play. Dena Cohen.
Alice in Dozenland. Wilhelmina Pitcher.
An Idea That Paid. Florence Miller Brooks.
Mathematical Nightmare. Josephine Skerrett.
Mathesis. Ella Brownell.
The Eternal Triangle. Gerald Raftery.
The Mathematics Club Meets. Wilimina Everett Pitcher.
The Case of "Matthews Mattix." Alice K. Smith.
More Than One Mystery. Celia Russell.

Price: 25¢ each.

Numbers and Numerals
A Story Book for Young and Old
Contents by Chapters

1. Learning to Count.
2. Naming the Numbers.
3. From Numbers to Numerals.
4. From Numerals to Computation.
5. Fractions.
6. Story of a few Arithmetic Words.
7. Mystery of Numbers.

The above monograph which will be the first of a series to be published soon by THE MATHEMATICS TEACHER will be sent postpaid to all subscribers whose subscriptions are paid up to November 1, 1936. The book will be sent postpaid to others at 25c each.

Annual Meeting of the National Council of Teachers of Mathematics

THE National Council of Teachers of Mathematics will hold their annual meeting in the Palmer House, Chicago, Illinois on February 19 and 20, 1937.

For classroom teachers and supervisors, the teachers of Chicago and vicinity are planning a mathematical exhibit to be placed in the foyer of the Grand Ball Room. Materials made by pupils and teachers which have been found useful in the teaching and learning of mathematics will be found there. There you will find suggestions for mathematical exhibits or a mathematical museum for your own school, games and puzzles based on mathematical principles, suggestions for pictures to be hung in mathematics classrooms, suggestions for books for mathematics libraries and many more things than can be mentioned in so limited a space.

For one of the meetings a moving-talking exhibit put on by real high school pupils and originally used to promote the interests of mathematics with the parent-teachers and also with the student body in high school assemblies will be shown. There will be other surprise features you must see and hear in order to appreciate them.

In the November issue of *The Mathematics Teacher* under *News Notes*, you will find an advance notice of the Discussion Luncheon scheduled for Saturday noon February 20. The discussion leaders and their topics are listed there and more leaders and topics are being supplied as the demand grows. An excellent opportunity for you to be heard at the meeting will be afforded. Any table will prove a gold mine, but if you can, **PUT IN YOUR RESERVATIONS EARLY.**

The opening meeting on Friday evening and the afternoon meeting on Saturday will be of a general nature of interest to every teacher of mathematics. On Saturday morning from 9:00 until 11:30 there will be three sectional meetings; one for the teacher of arithmetic, one for the problems of the junior and senior high school and the third for the teachers of junior college mathematics. A detailed program for all of these meetings, listing speakers and their topics will appear in the January issue of *The Mathematics Teacher*.

Chicago is centrally located and easily reached, traveling in any way you like, by motor, by train or by plane. Washington's Birthday, a school holiday for many, falls on the Monday following the convention thus extending the week end. We know that five hundred people will register for and attend these meetings. If even a small percentage of those who have never before attended an annual meeting consider the matter and plan for it, we can have seven hundred and fifty. If you are willing to call these meetings to the attention of the teachers and supervisors of arithmetic, the junior college teachers and any teachers of mathematics in your system who are not members of the Council and if you will invite them and urge them to attend with you, we can easily top one thousand. A meeting that is good for five hundred is better for one thousand. Bring your friends and **SPEND THE WEEK END OF FEBRUARY 19 WITH THE MATHEMATICS TEACHERS OF THE NATIONAL COUNCIL IN CHICAGO!**

EDITORIALS

The Position of the Classics

WRITING on "The Classical Outlook" Professor W. L. Carr recently said*

"The officers of the League realize, however, that these are times when the League must engage vigorously in certain activities which lie outside the field of research or service and in the field, one may almost say, of propaganda and defense. A teacher can not teach if his classes are legislated out of existence, and there are today in many states organized pressure groups determined to force this or that pet subject into an already crowded school curriculum with little regard for the subject or subjects which their innovations would displace. There is need for warning against those false prophets who would try to deceive administrative officers into believing that mere *change* is necessarily *progress* and against those eager enthusiasts who plead for breadth in our educational program but who do not seem to understand that any *solid* must have depth and length as well as breadth. Indeed, a good many of our educational theorists are themselves becoming alarmed at the way in which some of the educational experiments which they have inspired are working out in actual practice. They have seen, for example, that an overly sentimental application of the doctrine of interest may easily produce a generation of young men and women who lack the capacity to undertake or to stick to any task that does not happen to appeal to them at the moment. And they have seen that certain efforts at curriculum revision hopefully designed to give the pupil greater *breadth* of experience with life as it is lived today have succeeded only in giving him *no depth* and *no background*.

Now what is our duty and obligation as teachers of the Classics at a time like this? It seems to me that each of us is under a special obligation to take stock of his own objectives and materials and methods with a view to making his classroom more than ever before a center of stimulating and broadening and deepening experiences for each of his pupils. But that is not our whole duty in the days just ahead. Each of us must become an educational diplomat and seize every opportunity both in his own school system and in his state school system to oppose the almost fanatical attacks which are being directed at the Classics from certain quarters. And we must win as allies teachers of other subject-matter courses, especially teachers of the modern languages, of mathematics, and of the sciences, who are beginning to realize that they, too, are under fire."

While there may be something to be said against mathematics teachers being on the defensive, we are, to some extent at least, in much the same position in some schools today that the teachers of the Classics have been for several years. Unless we can develop a better type of mathematics education and learn to teach the subject in better and more useful ways the situation we are now in will get worse not better.

* October issue The Classical Outlook Vol. 14, p. 2.

The National Council Yearbooks

THE Yearbooks of the National Council of Teachers of Mathematics constitute a source of great help to teachers of mathematics all over this country and are beginning to be recognized by educational leaders as an outstanding contribution to educational literature. The tenth yearbook was voted one of the sixty-best educational books for the year 1935 and there is little doubt that the eleventh yearbook will fall in the 1936 list. Professor William McAndrew who reviewed the eleventh Yearbook in "School and Society" recently said that the book was a classic.

It would be a good idea if someone in the local county, state, or other organizations of mathematics teachers would take the responsibility of speaking about the yearbooks and would urge teachers to buy the complete set while they are still available as this chance will not be afforded much longer. The complete set may be obtained at a total cost of only $15.50 postpaid.

◆ IN OTHER PERIODICALS ◆

By Nathan Lazar
Alexander Hamilton High School, Brooklyn, New York

Algebra

1. Blake, Sue Avis. *Some of nature's curves. III.* School Science and Mathematics. 36: 717–21. October 1936.

The third in a series of articles on the occurence of well known curves in natural phenomena. The current instalment deals with the circle, the sphere, the parabola, the paraboloid, the catenary and the catenary of revolution. Five interesting photographs and one drawing are included.

2. Howland, Elizabeth Gray. *Methods of teaching the special products and their factors in ninth grade algebra.* School Science and Mathematics. 36: 771–76. October 1936.

A summary of the writer's unpublished Master's thesis, entitled "A Study of the Separate Versus the Together Method of Teaching the Special Products and Their Factors in Ninth Year Algebra." By the term "separate method" the writer refers to the "teaching procedure wherein the three special products are taught one after another and the factoring of each is then studied in turn. The term 'together method' is used to describe the method used when the teaching of each special product is followed immediately by the teaching of the factoring of that product before the study of the next product is begun."

After describing the materials and the technique used in the study, the author concludes as follows:

a. "Neither method seems to be consistently superior, either for finding the products or for factoring them. . . .

b. "While fewer process errors were made by the 'together' group on the final and long interval tests of products, the 'separate' method group made fewer process errors in those tests on factoring. Again, one gain offsets the other.

c. "Apparently not enough time elapses between the learning of the products and factoring them by the 'separate method' to invalidate the findings of educational psychology that teaching is most effective when as short a time as possible elapses between learning and application."

3. Johnston, L. S. *A note on partial fractions.* The American Mathematical Monthly. 43: 413–14. August–September, 1936.

An exposition of novel method of "resolving into its partial fractions, the proper fraction

$$\frac{f(x)}{(x^2+ax+b)\,(x^2+cx+d)}$$

where the denominator is not separated into linear factors." The writer found the method in the manuscript notes left by the late Rear Admiral John P. Merrell. "While it does not appear that the method possesses any advantage on the score of brevity, it is at least somewhat different from the more conventional methods."

4. *Minutiae in teaching procedure.* National Mathematics Magazine. 10: 175–77. February 1936.

The anonymous writer argues "that most of us could improve our teaching through some available record of cumulative experience of our colleagues." As an illustration he presents three methods of making the transition from the definition and numerical illustration of a determinant of the second order to its application of the solution of a pair of simultaneous equations.

Analytic Geometry and Calculus

1. Longley, W. R. *First exercise on differentials.* National Mathematics Magazine. 10: 219–26. March 1936.

A fairly complete account of an actual lesson on an important, perhaps crucial, concept in the calculus.

2. Temple, G. *The rehabilitation of differentials.* The Mathematical Gazette. 20: 120–131. May 1936.

A report of a paper and of the discussion that followed it at the annual meeting of the Mathematical Association, January 3, 1936, dealing with a topic of utmost importance to teachers of calculus.

The author admits that "at first sight, it appeared that it was impossible to construct a theory of differentials which should be both logical and teachable but happily it proved possible to cut this Gordian Knot and to formulate a theory of differentials which sacrifices neither logical rigor nor practical simplicity."

The outline of the paper follows:

I. Object of theory
II. The theory of differentials
III. Justification of theory
IV. Advantages of theory
a. Historical. b. Practical. c. Pedagogic.
V. Disadvantage of the theory
VI. Definition of circular and exponential functions.

3. Yates, Robert C. *Mechanically described curves.* National Mathematics Magazine. 10: 134–38. January 1936.

By means of diagrams and instructions the writer enumerates many curves that may be described without the tiresome labor of computing tables of values. The psychological advantage of such an approach to the teaching of analytic geometry are pointed out.

Miscellaneous

1. Betz, William. *The main purposes and objectives in teaching high school mathematics.* Bulletin of the Kansas Association of Mathematics Teachers. 10: 1–3. February 1936.

Abstract of an address at the meeting of the Mathematical Association of America and the National Council of Teachers of Mathematics, held at St. Louis, December 31, 1935.

2. Block, William E. *The "duomal" system of numeration and computation.* School Science and Mathematics. 36: 743–46. October 1936.

After enumerating the criteria underlying a desirable number system, the author points out the well-known difficulties usually encountered in teaching and learning the decimal system. He then proposes a "duomal system" which "combines the entirely simplified computational processes of the binary system with notational advantages of the octonary system."

Since it would be futile to attempt an adequate presentation of the nature of the proposed system in the space at our disposal, the interested reader is therefore urged to examine the original article.

3. Boyd, Paul P. *Mathematics as a personal experience.* National Mathematics Magazine. 10: 157–64. February 1936.

A very sensitive and illuminating analysis of the joys and pains, the benefits and dangers that have come to a mathematician in pursuit of his studies in his chosen field.

4. Christofferson, H. C. *Functional thinking and teaching in secondary school mathematics.* Bulletin of the Kansas Association of Mathematics Teachers. 10: 4–7. February 1936.

An abstract of a paper read January 1, 1936 at the St. Louis meeting of the National Council of Teachers of Mathematics.

5. Duncan, Dewey C. *Generalized Pythagorean numbers.* National Mathematics Magazine. 10: 209–11. March 1936.

The formulas for the finding of sets of "Pythagorean" or "right triangle" numbers are well-known. Thus to find *all* sets of integers such that

$$c^2 = a^2 + b^2$$

one merely has to use the three formulas

$$a = k(m^2 - n^2), \quad b = 2kmn, \qquad c = k(m^2 + n^2).$$

In this article the author exhibits "the most general formulas that yield n numbers, a_1, a_2, a_3, . . . a_n such that $a_1^2 + a_2^2 + a_3^2 + \ldots a_{n-1}^2 = a_n^2$."

The following set of four numbers will illustrate the meaning of the above:

$$2^2 + 6^2 + 9^2 = 11^2.$$

There are moreover 82 such sets of four numbers less than fifty. A table is included enumerating ~~all the~~ sets.

6. Hammond, Rolland B. *The value of mathematics from a student viewpoint.* Bulletin of the Kansas Association of Mathematics Teachers. 10: 3–4. February 1936.

An interesting exposition, by a senior in a high school, of what value mathematics has for him. He points out in detail how the knowledge of algebra and geometry helps in the formulation and in the comprehension of certain laws of sound and music.

7. Hopper, Grace Murray. *The ungenerated seven as an index to Pythagorean number theory.* The American Mathematical Monthly. 43: 409–13. August-September 1936.

As is well known, the Pythagorean Theorists have attributed quite fanciful attributes to various numbers. The most enigmatic of these is "the apotheosis of the number 7 as Athena, sprung full-armed from the head of Zeus." The author presents a very interesting and convincing explanation of the meaning of this traditional belief that was strongly intrenched but little understood.

8. Inglis, Alex. *Napier's education—a speculation.* The Mathematical Gazette. 20: 132–34. May 1936.

The author presents a plausible theory for the determination of the countries where Napier spent some years in study, but the identity of which is not known. This speculation affords an excellent example of the interplay of hypothesis and fact.

9. Miller, Walter M. *A discussion of the methods of science, history, art and mathematics.* National Mathematics Magazine. 10: 200–204. March 1936.

An acute analysis of the similarities and differences discoverable in the methods of science, history, art and mathematics.

10. Moulton, E. J. *Teaching mathematics.* National Mathematics Magazine. 10: 175. February 1936.

Trenchant comments on the teaching of mathematics. "Rarely are my questions designed merely to test the student's preparation or knowledge of the subject, but rather I attempt to be the leader in an exploring party in which all are participants."

11. Sleight, E. R. *The Scholar's Arithmetic.* National Mathematics Magazine. 10: 193–99. March 1936.

An interesting description of an arithmetic book that was popular in America in the years immediately following the Revolutionary War. After a short biography of the author of the text book, Daniel Adams, the writer gives a résumé of the contents of the book. Direct quotations succeed in conveying the spirit and philosophy of the author and of the era in which he lived.

12. *Work for the university entrance scholarships.* The Mathematical Gazette. 20: 73–87. May 1936.

A report of a discussion held at the Annual meeting of the Mathematical Association (England) on January 3, 1936.

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525 W. 120th Street, New York, N.Y.

NEWS NOTES

The twenty-first annual meeting of the Mathematical Association of America will be held at Durham and Chapel Hill, North Carolina, on Thursday, December 31, 1936 and Friday, January 1, 1937, in conjunction with the meeting of the American Mathematical Society.

Sessions of the Mathematical Association will be held Thursday afternoon at the University of North Carolina, Chapel Hill and Friday morning at Duke University, Durham. These are only nine miles apart. The program will consist of attractive papers by invited speakers; the complete program will be mailed to members early in December, as usual. The annual business meeting and election of officers will be held at the beginning of the session Friday morning.

The American Mathematical Society will hold regular sessions Tuesday and Wednesday at Duke University and Thursday morning at the University of North Carolina. Professor Solomon Lefschetz will deliver his retiring presidential address. By invitation of the Society's committee on program Professor J. M. Thomas will speak on the topic "Differential systems."

According to an announcement by the Society, a group of members of the Society who are interested in Topology are arranging a conference to be held Monday preceding the meetings of the Society. It is expected that the day will be spent mainly in informal discussion. It is suggested that all who are interested communicate with Professor G. T. Whyburn of the University of Virginia, Charlottesville, Va.

The joint dinner of the mathematicians will be held at Durham on Wednesday evening.

Through the generosity of Duke University, rooms in the dormitories on the Men's Campus will be available free of charge to members of the two mathematical organizations and their families. Meals will be served in the Men's Union at reasonable rates.

Rooms can also be had at the Durham hotels whose minimum rates are as follows: Washington Duke Hotel, single \$2.50, double with one bed \$3.50, double with twin beds \$4.50, all rooms having private bath. Hotel Malbourne, single \$1.75, double with one bed \$2.50, double with twin beds \$3.00, the price to be increased by 50 cents for private bath. The taxicab fare from either hotel to the campus is 25 cents.

For those who desire to stay in Chapel Hill, accommodations may be had at the Carolina Inn at the following rates: single rooms \$2.50 up, double rooms \$4.00 up, all rooms with bath.

Special buses will be provided for transportation to Chapel Hill on Thursday. An excursion to Pinehurst is tentatively planned for Wednesday afternoon, weather conditions permitting.

Railway rates are now low all over the country; members are advised to consult their local agents concerning holiday rates which may be available on the railroads.

The meeting of the Mathematics Section of the Western Zone of the New York State Teachers Association was held in Buffalo, N. Y. Friday, October 30, 1936. The program was as follows:

Chairman, Theresa L. Podmele, East High School, Buffalo, N. Y.

Place: School 17, Auditorium

Friday Morning, October 30

10:00 Topic: Experimental classes. One Year Course in General Mathematics

1. W. Price Aderman, Junior-Senior High School, Batavia, N. Y.
2. Elizabeth G. Benway, Hamburg Junior High School, Hamburg, N. Y.

10:30 Topic: The Modern Teacher of Arithmetic
Clara M. Dailey, State Normal School, Fredonia, N. Y.

11:00 Topic: Rationalizing Mathematics with Special Reference to Arithmetic
Genevieve Skehan, Whitney, School, No. 17, Rochester, N. Y.

Friday Afternoon, October 30

2:00 Business Meeting

2:15 Address: Demonstrative Geometry as a Basis for Critical Thinking
Harold P. Fawcett, Assistant Professor of Mathematics Education, Ohio State University, Columbus, Ohio.

The following resolution was adopted: Resolved that the Mathematics Section of the Western Zone of the New York State Teachers Association affiliate with the National Council of Teachers of Mathematics. Walden S. Cofran, principal of Jackson School, Batavia, N. Y. was elected chairman for next year.

A joint meeting of the Range Mathematics and Science Clubs was held at the O'Niel

Hotel in Chisholm October 15, 1936. The meeting was attended by sixty-five members of the organizations.

The program consisted of the numbers listed below:

Address of Welcome, Superintendent J. P. Vaughn, Chisholm

Response, H. G. Tiedeman, Mountain Iron

Solo Miss Marie DeLorimer Accompanied by Miss Tipton

Address "Mathematical Achievements of Ancient Peoples" by Dr. A. C. Piepkorn, Chisholm.

The speaker also showed three-reels of pictures taken by him in Iraq. These pictures showed the methods of work, manner of dress, and social customs of the people in Iraq. The pictures also conveyed the scientific methods necessary in order to do desirable excavating of ancient building and cities.

Short business meetings were held after the program.

Miss Evelyn Hoke and Mr. W. A. Porter, Chisholm High School were in charge of the program and arrangements.

The organizations will hold their next meeting in Mountain Iron during the first week in December.

H. S. Tiedeman, *Chairman*

A mathematics conference was held at State Teachers College in Indiana, Pennsylvania on October 10, 1936. The program follows:

9:50 General Meeting, Auditorium
Greetings, Dr. M. J. Walsh
The Place of Mathematics in Modern Education, Dr. W. D. Reeve

11:00 Conferences
Junior-Senior High School, Auditorium
Modern Trends in Mathematics
Round Table Discussion

Intermediate Grades, Wilson Hall
The training teachers in the fifth and sixth grades will discuss
1. The course of study for each grade
2. Major difficulties in fractions for fifth grade and decimal fractions for sixth grade
3. Objective materials found helpful for these topics

Primary Grades Wilson Hall
The training teachers in the first and second grades will discuss:
1. The arithmetic in each grade
2. Activities using arithmetic (Grade I)
3. Developing new combinations (Grade II)

12:00 Luncheon, College Dining Room

1:30 General Meeting, Auditorium
Miss Clara Shryock, Chairman
Topic: The Slow Moving Child in Arithmetic
1. General principles of Teaching, Dr. Richard Madden
2. Arithmetic in the Primary Grades, Miss Thelma Lessig
3. Arithmetic in Upper Grades, Mr. Leroy Schnell

2:30 Exhibit of Teaching Aids, Leonard Hall
Bulletin Boards, objective tests and materials

The mathematics section of the Maine State Teachers Association was held at Lewiston on October 29, 1936. The following program was rendered:

Chairman—Prin. Harold P. Andrews, Bridgton.

Vice Chairman—Mr. Frederick Richards, Camden.

Secretary—Mrs. Pauline B. Carter, Gardiner.

"Teaching Problems in High School Mathematics," Dr. William D. Reeve, Department of Mathematics, Teacher College, Columbia University.

"Some Devices to Make Mathematics a Human Study," Miss Margaret B. Jordan, Edward Little High School, Auburn.

"Creating Interest in Mathematics," Miss Eleanor Newman, Cony High School, Augusta.

Index photographed at the beginning for the convenience of the microfilm user.

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This organization has for its object the advancement of mathematics teaching in the elementary school and in junior and senior high schools. All persons interested in mathematics and mathematics teaching are eligible to membership. All members receive the official journal of the National Council—THE MATHEMATICS TEACHER—which appears monthly except June, July, August and September.

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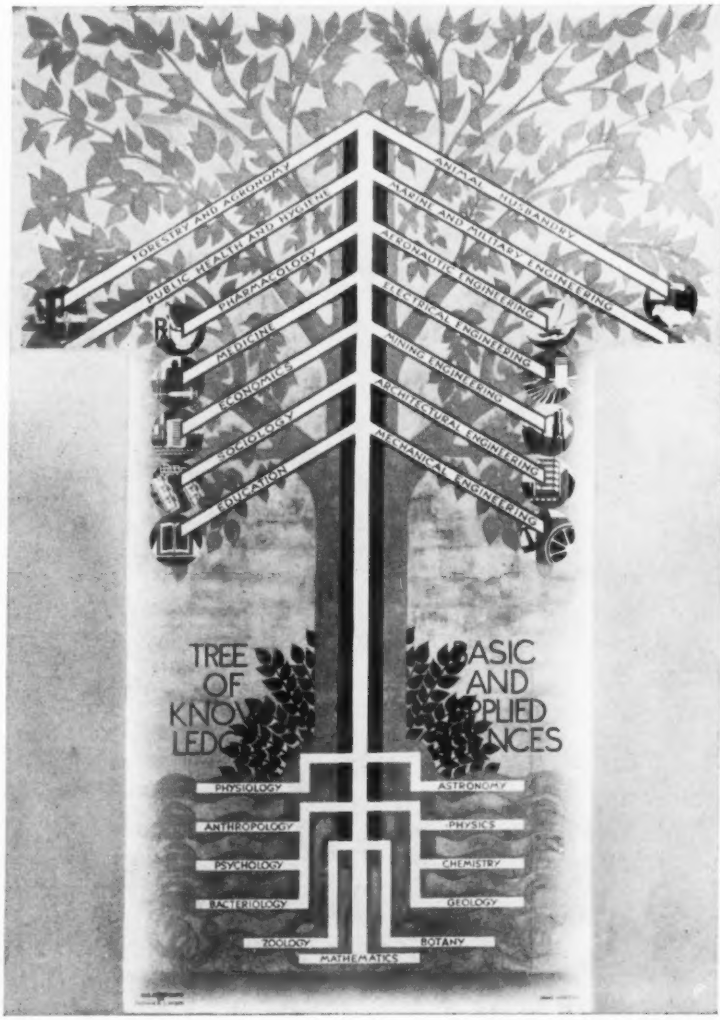

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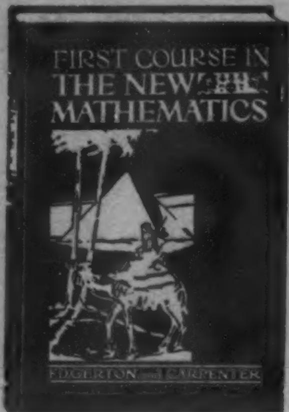

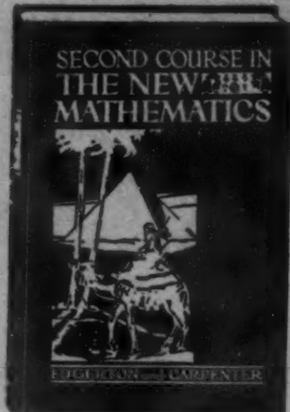

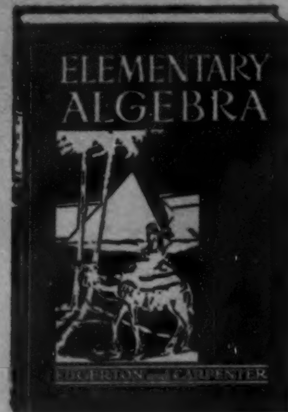

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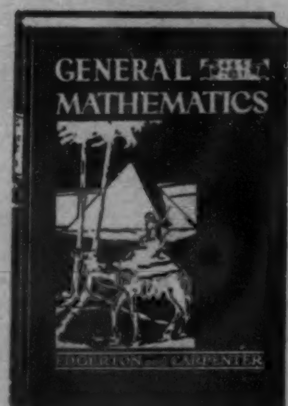

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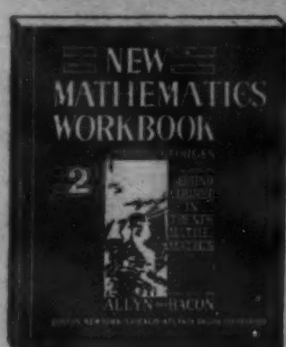

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